

EXTENSIVE FORM GAMES

Show all steps and calculations in your answers.

QUESTION 1. THE PIRATES AND THE SPOILS. There are n pirates, all ranked according to a strict hierarchy with the first pirate being the Captain, the second being the next in command, and so on and so forth down to the very last pitiful crewman. For simplicity, assume pirate 1 is the captain, 2 is the next in command, and so on down to n , who is the lowly crewman.

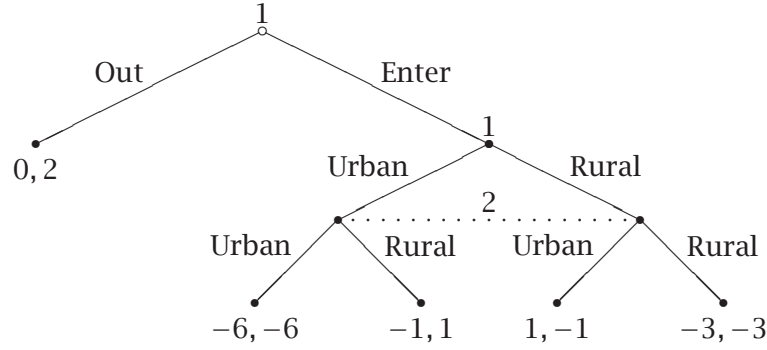
Presently, the pirates capture a merchant ship and have to decide how to divide the spoils consisting of $m > n$ gold coins. The procedure they use is as follows. The highest ranking pirate proposes some distribution of the coins. If at least half of the pirates agree to the proposal, it is implemented and the game ends. If, however, more than half the pirates fail to agree, then the proposer is made to walk the plank to his death and the next highest ranking pirate becomes the proposer. The procedure is then repeated. Pirates are self-preserving, greedy, and vindictive in that order: they most prefer to live, then to get as much gold coins as possible, and given both of these, to see as many superiors walk the plank as possible.

What is your guess about the Captain's predicament at the outset: is he in a strong position or is he screwed? (Write this down *before* solving the game.) What is the subgame perfect equilibrium of this game? How does this correspond to your prediction?

Answer. Let N denote the set of pirates. In order to secure the approval of his proposal, the Captain has to decide how to buy the support of at least half the pirates. So the questions are: which pirates to buy off and by how much. To calculate this, the Captain has to know the pirates' reservation values, that is, what they can guarantee themselves by rejecting his proposal.

We begin from the end (the longest terminal history). Suppose there are only two pirates left because all previous proposals have been rejected. Pirate $n - 1$ will simply propose to keep all m coins and because he votes for the proposal, it is accepted, leaving pirate n with 0 coins. Knowing this, pirate $n - 2$ can guarantee acceptance of his proposal by offering pirate n exactly 1 coin and keeping the other $m - 1$ for himself. Although pirate $n - 1$ votes against, the other two vote for the proposal and it is accepted. Knowing this, pirate $n - 3$ can also guarantee acceptance: he only needs to buy off $n - 1$, whose expected continuation payoff is 0 by offering him 1 gold coin and keeping the other $m - 1$ to himself. He and $n - 1$ vote for the proposal and it passes despite the Nay votes of the other two. Similarly, $n - 4$ can guarantee that his proposal is accepted. He needs the support of two other pirates, so he offers 1 gold coin to each of the two ($n - 2$ and n) whose continuation payoffs are 0, and proposes to keep $m - 2$ coins for himself. This proposal will also be accepted. Continuing this method all the way up to the Captain, we conclude that he needs the support of $k = \frac{n}{2}$ (if n is even) or $k = \frac{n+1}{2}$ (if n is odd) pirates, whose votes he can buy by offering them 1 gold coin each. Since he votes Yeah to his own proposal, the Captain distributes $k - 1$ gold coins to pirates $i \in \{N | i < k \wedge i \text{ odd}\}$ and keeps $m - k + 1$ coins for himself. The subgame perfect equilibrium is the profile where each pirate plays such a strategy. The outcome favors the Captain.

QUESTION 2. Consider the following incumbent-challenger game, in which once a challenger enters, both candidates must decide to campaign in rural areas or urban areas.



Find all Nash equilibria. Which ones are subgame perfect?

Answer. The strategic form and reduced strategic form of this game are given below.

	U	R
OU	0, 2	0, 2
OR	0, 2	0, 2
EU	-6, -6	-1, 1
ER	1, -1	-3, -3

	U	R
$\bar{O}$	0, 2	0, 2
ER	1, -1	-3, -3

There are two Nash equilibria in pure strategies, $\langle \bar{O}, R \rangle$ and $\langle ER, U \rangle$. What about mixed? Note that $\bar{O}$ strictly dominates EU for player 1, so in equilibrium he will never play EU . Player 1 will mix between $\bar{O}$ and ER whenever player 2 chooses U with probability $3/4$. Player 2 will only mix when 1 chooses $\bar{O}$ with probability 1. So there is a continuum of MSNE where player 1 plays $\bar{O}$ and player 2 mixes by playing U with probability strictly less than $3/4$.

The subgame beginning with player 1's second information set has two PSNE: $\langle R, U \rangle$ and $\langle U, R \rangle$. If the first PSNE is played in that subgame, player 1's expected payoff is 1, and so the optimal choice at his first information set is E . Hence, $\langle ER, U \rangle$ is an SPE. If the other PSNE is played, then player 1's expected payoff in the subgame is -1 , which means that his optimal choice at the first information set is O . Hence, $\langle OU, R \rangle$ is an SPE. The subgame also has a MSNE in which each player chooses U with probability $2/9$. Player 1's expected payoff in this equilibrium is

$$4/81(-6) + 49/81(-3) + 14/81(-1) + 14/81(1) < 0,$$

and so he will choose O at his first information set. Hence, $\langle (O, 2/9[U]), 2/9[U] \rangle$ is an SPE.

QUESTION 3. Consider a multi-stage game consisting of two players who play the Prisoner's Dilemma (from Figure 1) at the first stage, observe the outcome, then play the Prisoner's Dilemma again at the second stage, observe the outcome, then play the PD again, and so on, for a total of $N > 2$ stages.

		Player 2	
		C	D
Player 1	C	2,2	0,3
	D	3,0	1,1

Figure 1: The Prisoner's Dilemma

Each player's payoff is the sum of the payoffs from each repetition of the game. For example, if both players defected in all periods, each will get $N \times 1 = N$; if they defected in $K < N < \infty$ periods and cooperated in the remaining periods, each will get $K \times 1 + (N - K) \times 2$. Find the subgame perfect equilibria of this game. Is it possible to sustain the cooperative outcome in a *Nash equilibrium* of this game?

Answer. Because the game is finite, you can find the unique SPE by backward induction. In the last period, both players will defect (the only Nash equilibrium in that subgame). This implies that there is no reason not to defect in the previous period, and so on. The game will unravel, and defection at each stage is the SPE strategy for each player. To see that perpetual defection is the only Nash equilibrium of this game, you need to consider entire strategies (cannot backward induct). In equilibrium cooperation can only happen if the alternative is worse; that is, if defection is punished. The most deterrent threat involves the greatest punishment. This means that the most deterrent threat is a grim-trigger strategy that punishes a deviation by defecting in all remaining periods. If such a strategy cannot sustain cooperation, then neither will any more forgiving one (such as tit-for-tat). Suppose both players play grim trigger. Then the outcome is cooperation for all N periods, and each player gets a payoff $2N$. Can either one do better? Suppose player 1 switched to an alternative strategy: play grim trigger until the very last period, and then defect unconditionally there. Since 2 plays grim trigger, this would yield cooperation for $N - 1$ periods and the temptation payoff in the last period, or total payoff of $2(N - 1) + 3 = 2N + 1 > 2N$, so such a deviation will be profitable. Hence, both players using grim trigger cannot be Nash. What if both played grim trigger $N - 1$ periods and then defect unconditionally in the last? This cannot be Nash because either player can improve his payoff by preempting the defection in the penultimate period. Using an iterative argument like this, we can establish that no punishment strategy will be invulnerable to preemption, and hence profitable deviations always exist. Therefore, cooperation cannot be sustained in Nash equilibrium either.

QUESTION 4. Suppose in the Rubinstein bargaining model the players have different discount factors, δ_1 for player 1 and δ_2 for player 2. What is the stationary no-delay subgame perfect equilibrium of this game? What is the outcome? (You do not have to prove there are no SPE other than the stationary no-delay SPE you find.)

Answer. In a stationary no-delay SPE, player 2 (player 1) rejects all offers that do not give him at least the payoff from rejecting, and since player 1 (player 2) will never offer more than the minimum necessary to ensure acceptance, the following equations characterize the SPE offers:

$$\begin{aligned}\pi - x^* &= \delta_2 y^* \\ \pi - y^* &= \delta_1 x^*\end{aligned}$$

which has solutions $y^* = \pi(1 - \delta_1)/(1 - \delta_1\delta_2)$ and $x^* = \pi(1 - \delta_2)/(1 - \delta_1\delta_2)$. The unique stationary no-delay SPE is then the following strategy profile: player 1 always offers x^* , always accepts all $y \leq y^*$, and always rejects all $y > y^*$; player 2 always offers y^* , always accepts $x \leq x^*$, and always rejects

$x > x^*$, where x^* and y^* are the solutions above. The outcome is that agreement is reached immediately on the partition $(x^*, \pi - x^*)$, which is

$$\left(\frac{\pi(1 - \delta_2)}{1 - \delta_1\delta_2}, \frac{\delta_2\pi(1 - \delta_1)}{1 - \delta_1\delta_2} \right)$$

In other words, player 2's share is strictly smaller than the share he would obtain if he were to make the first offer. The proof that the strategies above are SPE is analogous to the one we did in class for the common discount factor, and so is not replicated here.

QUESTION 5. Two people take turns removing stones from a pile of n stones. Each person may, on each of his turns, remove either one stone or two stones. The person who takes the last stone is the winner and gets \$1 from the other person. Player 1 gets to move first. Who is the winner in the SPE for an arbitrary n ? Show all your work. (Hint: try solving the game for several small values of n and then prove the general result by induction.)

Answer. We find the SPE for $n = 1$ and $n = 2$, and then use these results to find the SPE for $n = 3$ and $n = 4$. If $n = 1$, player 1 removes one stone and wins is the only SPE. If $n = 2$, player 1 takes both stones and wins is the only SPE (because if he took only 1 stone, player 2 will take the last one and win). Thus, whoever gets to move first in any subgame with $n = 1$ or $n = 2$ is the winner.

Consider now $n = 3$. If player 1 removes one stone, player 2 will remove the remaining two and win. If player 1 removes two stones, player 2 will remove the remaining one and win. Therefore, the subgame with $n = 3$ has two subgame perfect equilibria: (i) player 1 removes one stone, player 2 removes two stones; and (ii) player 1 removes two stones, player 2 removes one stone. Player 2 wins in both cases.

Consider now $n = 4$. If player 1 removes one stone, the players will be in the subgame with $n = 3$ and player 2 as the first mover. This has two SPE, and player 1 wins in both. If player 1 removes two stones, the subgame is $n = 2$, which has a unique SPE with player 2 as the winner. Therefore, the subgame $n = 4$ has two SPE; (i) player 1 removes one stone, player 2 removes one stone, and player 1 removes two stones; and (ii) player 1 removes one stone, player 2 removes two stones, and player 1 removes one stone. In both SPE player 1 is the winner.

More generally, we note that player 1 wins in all SPE with $n = 1, 2, 4, 5, \dots$, while player 2 wins in all SPE with $n = 3, 6, \dots$. That is, if n is divisible by 3, then player 2 wins in all SPE. Otherwise, player 1 wins.

Let's prove the general claim by induction. We have already shown that it is true for $n = 1, 2, 3, 4$. Assume it is true for all integers through $n - 1$. We want to show that it is correct for n . First suppose that n is divisible by 3, and so the subgames following player 1's move have either $n - 1$ or $n - 2$ stones, neither of which is divisible by 3. From our hypothesis, player 2 wins in all SPE of these subgames. Thus, player 2 is the winner in every SPE of the whole game.

Suppose now that n is not divisible by 3. The two subgames are $n - 1$ and $n - 2$, as before, and one of them must be divisible by 3, in which case player 2 is the winner in the SPE of that subgame. This means that player 1 will choose the other subgame, where he is the winner in all SPE. Therefore, player 1 is the winner in every SPE of the whole game.

QUESTION 6. The firm Raider has entered the market that the firm Target has had for itself until now. Simultaneously, Raider and Target must choose a price strategy: high (H) or low (L). Subsequently, Raider chooses an advertising plan: If Raider observes that both firms have chosen the same price strategy, then he chooses between running ads or no ads. If the firms have chosen different price strategies, then there is no advertising.

The payoffs are as follows: if the firms choose different price strategies, the one that chose the low pricing gets 25 and the other gets 0. If both chose high prices, the payoffs are 0 without advertising, and (10, 5) with ads (where the first payoff is Raider's). If both chose low prices, then the payoffs are (−20, −30) without ads, and (0, −30) with ads.

Construct the extensive form of this game. Construct the strategic form and find all Nash equilibria. Which are subgame perfect?

Answer. The extensive form is given in Figure 2.

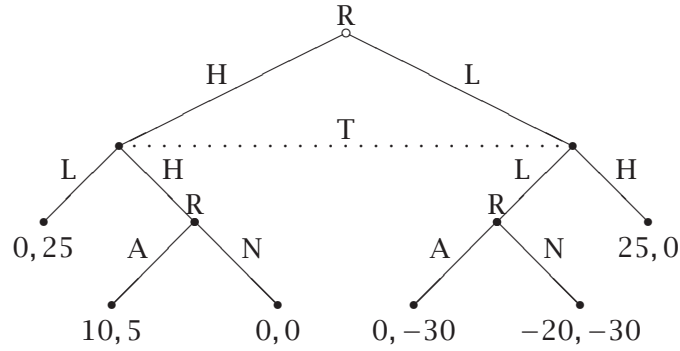


Figure 2: The Extensive Form.

Target has two pure strategies, $S_T = \{H, L\}$. Raider has three information sets. His pure strategies are:

$$S_R = \{HAA, HAN, HNA, HNN, LAA, LAN, LNA, LNN\}.$$

The strategic and reduced strategic forms are given in Figure 3. There are four Nash equilibria in pure

		Target	
		H	L
Raider	HAA	10, 5	0, 25
	HAN	10, 5	0, 25
	HNA	0, 0	0, 25
	HNN	0, 0	0, 25
	LAA	25, 0	0, −30
	LNA	25, 0	0, −30
	LAN	25, 0	−20, −30
	LNN	25, 0	−20, −30

		Target	
		H	L
Raider	HA	10, 5	0, 25
	HN	0, 0	0, 25
	LA	25, 0	0, −30
	LN	25, 0	−20, −30

Figure 3: The Strategic Forms.

strategies: $\langle LA, H \rangle$, $\langle LN, H \rangle$, $\langle HA, L \rangle$, and $\langle HN, L \rangle$. We shall let q denote the probability with which Target chooses H . We then have:

$$\begin{aligned} U_1(HA) &= 10q \\ U_1(HN) &= 0 \end{aligned}$$

$$\begin{aligned} U_1(LA) &= 25q \\ U_1(LN) &= 45q - 20. \end{aligned}$$

For any $q > 0$, LA strictly dominates both HA and HN , and so Raider will only be willing to mix between LA and LN . He would choose LA whenever $25q \geq 45q - 20 \Rightarrow q \leq 1$. That is, for any $q < 1$, Raider strictly prefers LA to LN , and in this case Target's unique best response is $q = 1$, so there is no mixed strategy equilibrium with $q < 1$. For $q = 1$, Raider is indifferent between LA and LN , and can therefore mix with probability $p \in [0, 1]$ for LA and $1 - p$ for LN . To any such strategy, Target's best response is $q = 1$. Therefore, there is a continuum of mixed-strategy Nash equilibria $\langle (0, 0, p, 1 - p), H \rangle$.

For $q = 0$, LN is strictly dominated (by any other pure strategy). Raider will be indifferent between HA , HN , and LA . Let p_1 and p_2 denote the probabilities of HA and HN , respectively, and let $1 - p_1 - p_2$ be the probability of LA . Because Target is choosing L , it must be the case that $5p_1 \leq 25p_1 + 25p_2 - 30(1 - p_1 - p_2)$, or $p_1 + 1.1p_2 \geq \frac{3}{5}$. (That is, the probability of LA is sufficiently low.) Hence, when this condition is met, L is the best response. There is a continuum of mixed-strategy Nash equilibria $\langle (p_1, p_2, 1 - p_1 - p_2, 0), L \rangle$ where $p_1 + 1.1p_2 \geq \frac{3}{5}$.

To find the subgame perfect equilibria, note that regardless of whether both players chose H or both chose L , Raider's best last move is to choose A . This produces the reduced strategic form in Figure 4.

		Target	
		H	L
Raider	H	10, 5	0, 25
	L	25, 0	0, -30

Figure 4: The Subgame Perfect Strategic Form.

This game has two pure-strategy Nash equilibria, $\langle L, H \rangle$ and $\langle H, L \rangle$. Again, let q denote the probability with which Target chooses H , and let p denote the probability with which Raider chooses H . Raider will only be willing to mix if $10q = 25q$, which implies $q = 0$. Target will choose $q = 0$ whenever $5p \leq 25p - 30(1 - p)$, or whenever $p \geq \frac{3}{5}$. Hence, there is a continuum of mixed-strategy Nash equilibria $\langle (p, 1 - p), L \rangle$ with $p \in [\frac{3}{5}, 1]$.

The game has two pure-strategy subgame perfect equilibria, $\langle HAA, L \rangle$ and $\langle LAA, H \rangle$, and a continuum of mixed-strategy SPE $\langle pAA, L \rangle$, where $p \in [\frac{3}{5}, 1]$ is the probability of H at Raider's first information set.