

STRATEGIC FORM GAMES

Show all steps and calculations in your answers, for proofs justify each step.

QUESTION 1. Consider the Rock, Paper, Scissors Game and determine $BR_1(\sigma_2)$ for the following mixed strategies for player 2: (a) $\sigma_2 = (1, 0, 0)$; (b) $\sigma_2 = (1/6, 1/3, 1/2)$; (c) $\sigma_2 = (1/2, 1/4, 1/4)$; and (d) $\sigma_2 = (1/3, 1/3, 1/3)$.

Answer. Let (r, p, s) denote a mixed strategy for player 1, where p is the probability assigned to P , r is the probability assigned to R , and $s = 1 - p - r$ is the probability assigned to S . The best-response correspondence must specify values for r and p .

$$BR_1(1, 0, 0) = \{(0, 1, 0)\}$$

$$BR_1(1/6, 1/3, 1/2) = \{(r, 0, s) \mid r, s \in [0, 1] \wedge r + s = 1\}$$

$$BR_1(1/2, 1/4, 1/4) = \{(0, 1, 0)\}$$

$$BR_1(1/3, 1/3, 1/3) = \{(r, p, s) \mid r, p, s \in [0, 1] \wedge r + p + s = 1\}$$

QUESTION 2. Consider the game in Figure 1. Let x denote the probability that player 1 chooses U , and y denote the probability that player 2 chooses L . Derive the best responses graphically by plotting player i 's payoff to his two pure strategies as a function of his opponent's mixed strategy. Plot the two reaction correspondences in the (x, y) space. What are the Nash equilibria?

		Player 2	
		L	R
Player 1	U	1, -1	3, 0
	D	4, 2	0, -1

Figure 1: The Plotting Game.

Answer. First, express the payoffs as functions of the opponent's mixed strategy:

$$u_1(U, y) = 3 - 2y$$

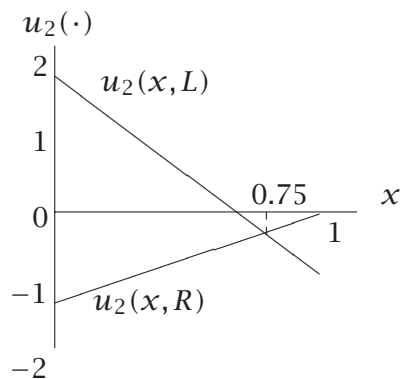
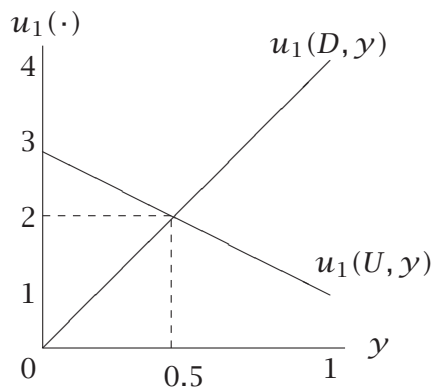
$$u_2(x, L) = 2 - 3x$$

$$u_1(D, y) = 4y$$

$$u_2(x, R) = x - 1$$

Derive the best responses graphically by simply plotting the two pairs of functions:

We obtain the intersections by setting $3 - 2y = 4y$, which yields $y = .5$, and setting $2 - 3x = x - 1$, which yields $x = .75$. We can now plot the best responses in the (x, y) space by noting the following (obvious



from the plots above):

$$BR_1(y) = \begin{cases} 1 & \text{if } y < .5 \\ [0, 1] & \text{if } y = .5 \\ 0 & \text{if } y > .5 \end{cases} \quad BR_2(x) = \begin{cases} 1 & \text{if } x < .75 \\ [0, 1] & \text{if } x = .75 \\ 0 & \text{if } x > .75 \end{cases}$$

The plot we're after is shown in Figure 2. Let (x, y) denote a mixed strategy profile. There are three Nash equilibria: $(0, 1)$, $(1, 0)$, and $(.75, .5)$. The first two are in degenerate mixed strategies, and the last one is in completely mixed strategies. (Of course, you could have found the two pure strategy Nash equilibria just by inspecting the payoff matrix.)

QUESTION 3. THE CHAIRMAN'S PARADOX. A committee consists of three players, 1, 2, and 3, with player 1 elected as chairman. The committee must select one of three alternatives, A, B, and C. Members vote simultaneously for an alternative, abstaining is not allowed. The alternative with the most votes wins. If no alternative receives a majority, then the chairman selects his most favored alternative. The payoff functions are

$$\begin{aligned} u_1(A) &= u_2(B) = u_3(C) = 2, \\ u_1(B) &= u_2(C) = u_3(A) = 1, \\ u_1(C) &= u_2(A) = u_3(B) = 0. \end{aligned}$$

This game has three equilibrium *outcomes*: A, B, and C. However, it has more *equilibria* than this. Find all pure-strategy Nash equilibria of this game. Apply iterated elimination of weakly dominated strategies to see why this game is called the way it is. Explain.

Answer. You could write down the three 3×3 payoff matrices and find the Nash equilibria by inspection but you don't have to. First note that no profile in which the tie breaking rule is used can be an equilibrium: since the outcome is A, player 2 (whose payoff in this case is 0) can switch to voting either B or C, depending on the profile, and improve his payoff.

Also note that all profiles, in which all three players vote for the same alternative are Nash equilibria: given that the other two players are choosing the alternative, it receives a majority and wins, and the

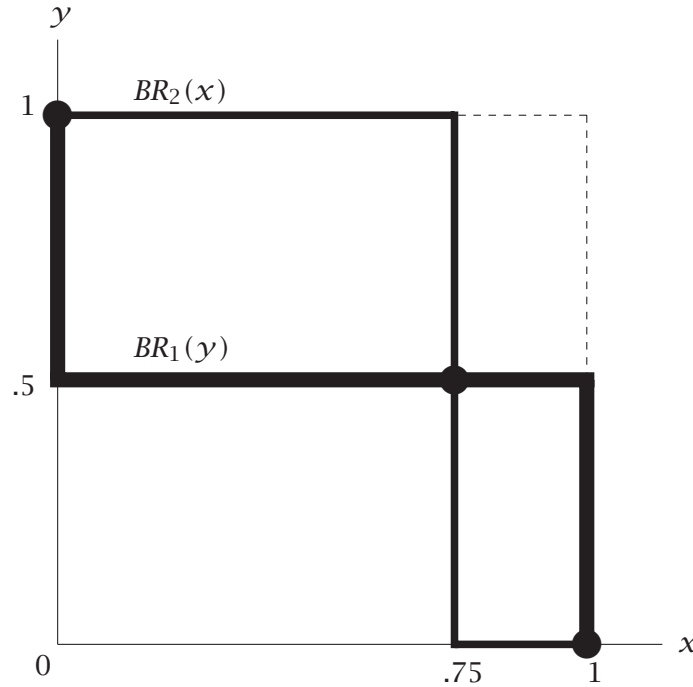


Figure 2: Best Responses in the Plotting Game.

third player cannot change that, so he may as well vote for it. So (A, A, A) , (B, B, B) , and (C, C, C) are all Nash equilibria.

Are there other ways for alternative A to win in equilibrium? Not unless player 1 votes for A for otherwise the supporter of the alternative for which 1 is voting can switch from A to whatever 1 is voting for and ensure that his most preferred alternative will win. So, we consider only profiles where 1 is voting A. The profile (A, B, A) is a Nash equilibrium because 2 cannot change the outcome and while player 3 can, this would only make him worse off. The profile (A, A, B) is *not* an equilibrium because player 2 can profitably switch to B.

Are there other ways for alternative B to win in equilibrium? No: unless two players are voting for B, the third one can change the outcome to something he likes more.

Are there other ways for alternative C to win in equilibrium? Yes: if both player 2 and player 3 vote C the chairman can vote A and no one can benefit: player 3 is getting his most preferred outcome and player 2 can only change the outcome to A by voting either A or tying with B, in which case he gets his worst payoff. So (A, C, C) is another Nash equilibrium.

The game thus has 5 Nash equilibria in pure strategies. If you apply IEWDS, you get:

- A weakly dominates B for player 1
- B weakly dominates A for player 2
- C weakly dominates A for player 3
- A weakly dominates C for player 1

- C weakly dominates B for player 3
- C weakly dominates B for player 2

The IEWDS equilibrium is (A, C, C) , where the chairman gets his *worst* preferred outcome! The problem with the equilibria in which player 1 gets his most preferred outcome (A, A, A) and (A, B, A) , involve both player 2 and player 3 playing weakly dominated strategies in the first one and player 3 playing a weakly dominated strategy in the second one. Thus, IEWDS will always eliminate these equilibria. In addition, since (B, B, B) involves a weakly dominated strategy by player 1, it will also get eliminated, leaving only the equilibria with the worst outcome, one of which (the one above) will survive the elimination (the other is (C, C, C) but it involves a weakly dominated strategy by player 1).

You might have expected that player 1 is in an enviable position since it appears that he has quite a bit of power when he can decide ties. But as we saw, ties never occur in equilibrium, so this power is “meaningless.” What’s worse: it gives the other two players incentives that are detrimental to player 1.

QUESTION 4. Show that the two-player game in Figure 3 has a unique equilibrium. (Hint: Show that it has a unique pure strategy equilibrium. Then show that player 1 cannot put positive weight on any combination of pure strategies.)

		Player 2		
		L	C	R
Player 1	U	1, -2	-2, 1	0, 0
	M	-2, 1	1, -2	0, 0
	D	0, 0	0, 0	1, 1

Figure 3: The Unique Equilibrium Game.

Answer. The pure strategy Nash equilibrium is (D, R) , which you can get by simple inspection of the payoff matrix. In order to show that it is unique, we make good use of the liberal hint.

Suppose that in some mixed strategy equilibrium player 1 puts positive weight on both U and M . We now show that this implies that player 2 must put equal weight on L and C . Since player 1 is putting positive weight on both U and M in equilibrium, the payoffs from the two pure strategies are the same given player 2’s equilibrium strategy:

$$u_1(U, \sigma_2) = \sigma_2(L) - 2\sigma_2(C) = -2\sigma_2(L) + \sigma_2(C) = u_1(M, \sigma_2)$$

Solving this equation yields $\sigma_2(L) = \sigma_2(C) = \hat{\sigma}_2$, so player 2 puts equal weight on L and C . Suppose $\hat{\sigma}_2 > 0$, and so

$$u_1(\sigma_1, \sigma_2) = -\hat{\sigma}_2[\sigma_1(U) + \sigma_1(M)] + \sigma_1(D)(1 - 2\hat{\sigma}_2) < u_1(D, \sigma_2) = 1 - 2\hat{\sigma}_2$$

where the right hand side of the inequality is player 1’s payoff if he simply chooses D instead of mixing. Since player 1 can do better by doing so, this contradicts the equilibrium assumption.

Therefore, $\hat{\sigma}_2 = 0$, and so player 2 chooses R , in which case player 1’s unique best response is D , contradicting the assumption of positive weights on U and M . We conclude that in equilibrium player 1 will not put positive weights on both U and M .

We now have to eliminate other possibilities: player 1 puts positive weights (a) on both U and D but not M , or (b) on both M and D but not U . Since these two cases are analogously proven, we shall just show the proof for case (a).

Suppose that in some mixed strategy equilibrium player 1 puts positive weight on both U and D but not M . Since player 1 does not choose M , L is strictly dominated by R , and so $\sigma_2(L) = 0$ in equilibrium.

But if this is the case, D strictly dominates U , and $\sigma_1(U) = 0$ in equilibrium as well, contradicting the assumption that player 1 puts positive probability on both U and D .

Using the same procedure, we conclude that player 1 cannot put positive probability on both M and D but not U in equilibrium either.

Because this exhausts the possible mixtures for player 1, it follows that in equilibrium player 1 only puts positive weight on D . Since player 2's unique best response to this is to play R , we conclude that the pure strategy Nash equilibrium (D, R) is indeed the unique Nash equilibrium of this game.

QUESTION 5. Find all mixed strategies of player 1 that strictly dominate U in Figure 4.

		Player 2	
		L	R
Player 1	U	1, 3	1, 0
	M	4, 0	0, 3
	D	0, 1	3, 1

Figure 4: Pure Strategy Dominated by a Mixed Strategy.

Answer. First note that there are no degenerate mixed strategies that strictly dominate U (neither M nor D strictly dominates it). Here are the various supports that we should look at:

$$\text{supp}(\sigma_a) = \{U, M, D\}$$

$$\text{supp}(\sigma_b) = \{U, M\}$$

$$\text{supp}(\sigma_c) = \{U, D\}$$

$$\text{supp}(\sigma_d) = \{M, D\}$$

A mixed strategy strictly dominates a pure strategy if it does better regardless of what the opponent does. This means that it does better against all pure strategies of the opponent (and if this is so, it will also do better against any mixed strategies as well). We therefore only need to compare whether a mixed strategy does better both against L and against R . Since $u_1(U, \cdot) = 1$, we must find mixed strategies whose expected payoff is strictly greater than 1 against both L and R .

We begin with the mixed strategies σ_b and σ_c and note that they cannot strictly dominate U . The first one yields a payoff strictly less than 1 against R for any positive probability put on M . The second one yields a payoff strictly less than 1 against L for any positive probability put on D .

Let's now look at σ_d . We must find $\sigma_d(M)$ such that $u_1(\sigma_d, L) > 1$ and $u_1(\sigma_d, R) > 1$ (noting that since $\sigma_d(D) = 1 - \sigma_d(M)$ we don't have to specify it explicitly):

$$u_1(\sigma_d, L) = 4\sigma_d(M) > 1 \quad \Rightarrow \sigma_d(M) > 1/4$$

$$u_1(\sigma_d, R) = 3(1 - \sigma_d(M)) > 1 \quad \Rightarrow \sigma_d(M) < 2/3$$

Therefore, any mixed strategy σ_1 with $\sigma_1(M) \in (1/4, 2/3)$ and $\sigma_1(D) = 1 - \sigma_1(M)$ will strictly dominate U .

Finally we turn to σ_a , the fully mixed strategy. We make use of the result for σ_d : Since σ_d strictly dominates U , any completely mixed strategy that assigns positive probability to σ_d will also strictly dominate U . That is, for any $p \in (0, 1)$, the completely mixed strategy σ_1 with $\sigma_1(U) = 1 - p$, $\sigma_1(M) = px$ where $x \in (1/4, 2/3)$, and $\sigma_1(D) = p(1 - x)$ will strictly dominate U .

Let's do a quick check by letting $p = 1/3$ and $x = 1/2$. Then

$$u_1(\sigma_1, L) = (1)(2/3) + (4)(1/3)(1/2) = 4/3 > 1$$

$$u_1(\sigma_1, R) = (1)(2/3) + (3)(1/3)(1 - 1/2) = 7/6 > 1$$

I hope you noticed that player 2's payoffs are completely irrelevant for determining which of player 1's strategies are strictly dominated.

QUESTION 6. Construct a 2-player 2×2 simultaneous move game where a Nash equilibrium does not survive iterated elimination of weakly dominated strategies. Show the order of elimination that “loses” the equilibrium.

Answer. The following game presents an example, where the Nash equilibrium fails to survive iterated elimination of weakly dominated strategies:

		Player 2	
		<i>L</i>	<i>R</i>
Player 1	<i>U</i>	0,0	1,0
	<i>D</i>	0,1	3,3

Figure 5: Nash Equilibrium Fails to Survive IEWD Strategies.

There are two pure strategy Nash equilibria, (U, L) and (D, R) . However, eliminating U because it is weakly dominated by D also loses the first Nash equilibrium.

QUESTION 7. Find all Nash equilibria of the three-player game in Figure 6, where player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices (matrix on the left is strategy A and matrix on the right is strategy B). Show your work for all possible combinations of fully mixed, partially mixed, and pure strategy profiles.

	A			B	
	<i>A</i>	<i>B</i>		<i>A</i>	<i>B</i>
<i>A</i>	1, 1, 1	0, 0, 0	<i>A</i>	0, 0, 0	0, 0, 0
<i>B</i>	0, 0, 0	0, 0, 0	<i>B</i>	0, 0, 0	4, 4, 4

Figure 6: The Three-Player Game.

Answer. There are two Nash equilibria in pure strategies, (A, A, A) and (B, B, B) . Let's find the MSNE in mixed strategies. Let $\sigma_1(A) = p$, $\sigma_2(A) = q$, and $\sigma_3(A) = r$, with $p, q, r \in [0, 1]$. If players 1 and 2 follow their mixed strategies, player 3 gets $u_3(p, q, A) = pq$ from pure strategy A and $u_3(p, q, B) = (4)(1 - p)(1 - q)$ from pure strategy B. In a MSNE players are indifferent between their pure strategies, so

$$pq = (4)(1 - p)(1 - q) \Rightarrow p = \frac{4(1 - q)}{4 - 3q}.$$

Using the same logic for the other two players, we also find that

$$qr = 4(1 - r)(1 - q) \Rightarrow q = \frac{4(1 - r)}{4 - 3r}$$

$$pr = 4(1 - p)(1 - r) \Rightarrow p = \frac{4(1 - r)}{4 - 3r}$$

and the above three equations imply

$$p = q = \frac{4(1 - q)}{4 - 3q}$$

This results in a quadratic, $3q^2 - 8q + 4 = 0$, which we now solve for q . The discriminant is $D = b^2 - 4ac = 16$, and so the quadratic has two solutions, $q_1 = 2$ and $q_2 = 2/3$. But the first is not a valid probability, so the only solution is $q = 2/3 = p = r$.

Let's verify that these mixed strategies are indeed Nash equilibrium. Player 3's expected payoff is

$$u_3(2/3, 2/3, 2/3) = 8/27 + (4)1/27 = 12/27 = 4/9$$

If the player deviates to A, the payoff is $u_3(2/3, 2/3, A) = 4/9$, so it is not profitable. If he deviates to B, the payoff is $u_3(2/3, 2/3, B) = 4/9$, which is not profitable either. Of course, we should have expected this result because we explicitly solved for his opponents' mixed strategies that make player 3 indifferent between his pure strategies.

It is actually quite easy to see that there is no MSNE in which some player plays a pure strategy. Suppose $\sigma_1(A) = 1$ and $\sigma_3(A) \in [0, 1]$. What is player 2 to do? Since $u_2(1, A, \sigma_3(A)) = \sigma_3(A)$ and $u_2(1, B, \sigma_3(A)) = 0$, for any $\sigma_3(A) > 0$ player 2 will strictly prefer to play A. But in this case player 3 definitely will play A as well, so the equilibrium is in pure strategies. If $\sigma_3(A) = 0$, on the other hand, player 2 is indifferent and can mix with some $\sigma_2(A) \in [0, 1]$.

However, if $\sigma_3(A) = 0$, then for any $\sigma_2(A) < 1$ layer 1 would go B for sure, contradicting the assumption that $\sigma_1(A) = 1$. So it must be the case that $\sigma_2(A) = 1$. Since $\sigma_3(A) = 0$, player 1 is indifferent between his pure strategies and can mix with any probability.

However, for any $\sigma_1(A) > 0$ player 3 will play A whenever $\sigma_2(A) = 1$ but since $\sigma_3(A) = 0$, it must be the case that $\sigma_1(A) = 0$ as well. But this contradicts the assumption that $\sigma_1(A) = 1$. Hence, there can be no Nash equilibrium in which some player plays a pure strategy while other players mix.