

PREFERENCES AND EXPECTED UTILITY THEORY

Show all steps and calculations in your answers, for proofs justify each step.

QUESTION 1. Assume that $\succ$ is asymmetric and negatively transitive. Prove that:

- (a) $\succ$ is irreflexive, transitive (by contradiction), and acyclic.
- (b) $\sim$ is reflexive and transitive.
- (c) if $w \sim x$, $x \succ y$, and $y \sim z$, then $w \succ y$ and $x \succ z$.

Answer. Since we have proven that $\succ$ is complete if it is asymmetric and negatively transitive in class, we shall use it if needed without a new proof.

(a) Three things to prove:

- Prove that $\succ$ is irreflexive: $x \succ x$ for no x . Pick an arbitrary x , and label (a) $y = x$.
 1. $\neg[[x \succ y] \wedge [y \succ x]]$ asymmetry
 2. $\neg[x \succ y] \vee \neg[y \succ x]$ (1), DeMorgan
 3. $\neg[x \succ x] \vee \neg[x \succ x]$ (2) and (a)
 4. $\neg[x \succ x]$ (3), tautology
 5. $\forall x : \neg[x \succ x]$ (4), universal generalization.
- Prove that $\succ$ is transitive. Assume (a) $x \succ y$, and (b) $y \succ z$. Show $x \succ z$.
 1. $\neg[z \succ y]$ (b), asymmetry
 2. $[x \succ z] \vee [z \succ y]$ (a), negative transitivity
 3. $x \succ z$ (1) and (2), disjunctive syllogism
- Prove that $\succ$ is acyclic. We already know that it is transitive, so:
 1. $x_1 \succ x_2$ premise
 2. $x_2 \succ x_3$ premise
 - $\vdots$
 - $n - 1.$ $x_{n-1} \succ x_n$ premise
 - $n.$ $x_1 \succ x_n$ (1)-(n-1), transitivity
 - $n + 1.$ $x_1 \neq x_n$ (n), asymmetry

(b) Two things to prove:

- $\sim$ is reflexive:
 1. $x = y$ premise
 2. $\neg[x \succ y]$ (1), irreflexivity of $\succ$
 3. $y \geq x$ (2), definition
 4. $\neg[y \succ x]$ (1), irreflexivity of $\succ$
 5. $x \geq y$ (4), definition
 6. $[x \geq y] \wedge [y \geq x]$ (3) and (5), conjunction
 7. $x \sim y$ (6), definition
 8. $x \sim x$ (7) and (1)
- $\sim$ is transitive:

1.	$x \sim y$	premise
2.	$y \sim z$	premise
3.	$[x \succeq y] \wedge [y \succeq x]$	(1), definition
4.	$x \succeq y$	(3), simplification
5.	$[y \succeq z] \wedge [z \succeq y]$	(2), definition
6.	$y \succeq z$	(5), simplification
7.	$x \succeq z$	(4) and (6), transitivity of $\succeq$
8.	$y \succeq x$	(3), simplification
9.	$z \succeq y$	(5), simplification
10.	$z \succeq x$	(8) and (9), transitivity of $\succeq$
11.	$[x \succeq z] \wedge [z \succeq x]$	(7) and (10), conjunction
12.	$x \sim z$	(11), definition

(c) Assume (a) $w \sim x$, (b) $x \succ y$, and (c) $y \sim z$. Prove $[w \succ y] \wedge [x \succ z]$ (by contradiction):

1.	$\neg[w \succ y] \wedge \neg[x \succ z]$	assumed premise
2.	$\neg[w \succ y]$	(1), simplification
3.	$y \succeq w$	(2), definition
4.	$\neg[y \succ x]$	(b), asymmetry
5.	$x \succeq y$	(4), definition
6.	$[w \succeq x] \wedge [x \succeq w]$	(a), definition
7.	$w \succeq x$	(6), simplification
8.	$w \succeq y$	(5) and (7), transitivity
9.	$[w \succeq y] \wedge [y \succeq w]$	(3) and (8), conjunction
10.	$y \sim w$	(9), definition
11.	$x \sim y$	(10) and (a), transitivity
12.	$\neg[y \succeq w]$	RRA: (11) contradicts (b)
13.	$w \succ y$	(8) and (12), definition
14.	$\neg[x \succ z]$	(1), simplification
15.	$z \succeq x$	(14), definition
16.	$[y \succeq z] \wedge [z \succeq y]$	(c), definition
17.	$y \succeq z$	(16), simplification
18.	$[x \succeq y] \wedge \neg[y \succeq x]$	(b), definition
19.	$x \succeq y$	(18), simplification
20.	$x \succeq z$	(17) and (19), transitivity
21.	$[z \succeq x] \wedge [x \succeq z]$	(15) and (20), conjunction
22.	$x \sim z$	(21), definition
23.	$x \sim y$	(c) and (22), transitivity
24.	$\neg[z \succeq x]$	RRA: (23) contradicts (b)
25.	$x \succ z$	(20) and (24), definition
26.	$[w \succ y] \wedge [x \succ z]$	(13) and (25), conjunction

QUESTION 2. Suppose $X = \{\$0, \$2 \times 10^6, \$10 \times 10^6\}$. Consider the following four lotteries: $L_1 = (0, 1, 0)$, $L_2 = (.01, .89, .1)$, $L_3 = (.89, .11, 0)$, and $L_4 = (.9, 0, .1)$. An individual tells you she prefers L_1 to L_2 , and L_4 to L_3 . Can her preferences be represented by a function with the expected utility form? How would you rank these lotteries (remember that we're talking about millions of dollars)? Can you represent your preferences with an expected utility function?

Answer. So we know that the decision-maker's preferences are:

$$L_1 \succ L_2 \quad \text{and} \quad L_4 \succ L_3.$$

Suppose that her preferences are represented by a function $u(\cdot)$ with the expected utility form $U(\cdot)$. Then we would have:

$$\begin{aligned} U(L_1) > U(L_2) &\Leftrightarrow u(2 \times 10^6) > 0.01u(0) + 0.89u(2 \times 10^6) + 0.1u(10 \times 10^6) \\ &\Leftrightarrow 0.11u(2 \times 10^6) > 0.01u(0) + 0.1u(10 \times 10^6) \\ U(L_4) > U(L_3) &\Leftrightarrow 0.9u(0) + 0.1u(10 \times 10^6) > 0.89u(0) + 0.11u(2 \times 10^6) \\ &\Leftrightarrow 0.11u(2 \times 10^6) < 0.01u(0) + 0.1u(10 \times 10^6), \end{aligned}$$

which is impossible, and we have reached a contradiction. Therefore, these preferences cannot be represented by a function with the expected utility form. What's interesting about this example is that in experimental studies people often express such preferences. That is, they often prefer getting \$2 million with certainty to playing a lottery where there is 1% chance that they would get nothing, an 89% chance of \$2 million, and 10% chance of \$10 million. In this case they display aversion to an increase of 1% in the risk of getting nothing. But the same people then also prefer a 10% chance of \$10 million and 90% chance of nothing to a lottery that would give them 11% chance of \$2 million and 89% chance of nothing. In this case they display aversion to a decrease of 1% in the risk of getting nothing. This is not to say that such people are irrational. We are just saying that we cannot represent their preferences with an expected utility function.

QUESTION 3. There are three possible outcomes; in the outcome a_i the decision-maker gains $\$a_i$, where $a_1 < a_2 < a_3$. The decision-maker prefers a_3 to a_2 to a_1 , and she prefers the lottery $p = (0.3, 0, 0.7)$ to $q = (0.1, 0.4, 0.5)$ to $r = (0.3, 0.2, 0.5)$ to $s = (0.45, 0, 0.55)$. Is this information consistent with the decision-maker's preferences being represented by the expected value of a payoff function? If so, find a payoff function consistent with the information. If not, show why not. Answer the same questions when, alternatively, the decision-maker prefers the lottery $p_1 = (0.4, 0, 0.6)$ to $q_1 = (0, 0.5, 0.5)$ to $r_1 = (0.3, 0.2, 0.5)$ to $s_1 = (0.45, 0, 0.55)$.

Answer. We have the following preferences:

$$\begin{aligned} a_3 &> a_2 > a_1 \\ (.3, 0, .7) &> (.1, .4, .5) > (.3, .2, .5) > (.45, 0, .55) \end{aligned}$$

We want to know if there exists some utility function, whose expected value can represent these preferences? In other words, we want to find $u(\cdot)$, such that:

$$\begin{aligned} U(p) &= (.3)u(a_1) + (0)u(a_2) + (.7)u(a_3) \\ &> U(q) = (.1)u(a_1) + (.4)u(a_2) + (.5)u(a_3) \\ &> U(r) = (.3)u(a_1) + (.2)u(a_2) + (.5)u(a_3) \\ &> U(s) = (.45)u(a_1) + (0)u(a_2) + (.55)u(a_3). \end{aligned}$$

From the second and the third expected utilities, we conclude that $u(a_1) < u(a_2)$; and from the first and the third, we conclude that $u(a_2) < u(a_3)$. Therefore, $u(a_1) < u(a_2) < u(a_3)$. This ensures that $U(q) > U(r)$ and $U(p) > U(r)$. It is sufficient to check whether $U(p) > U(q)$ and $U(r) > U(s)$. Note that $U(p) > U(q)$ yields $u(a_1) + u(a_3) > .2u(a_2)$, which holds because $u(a_3) > u(a_2)$. Similarly, $U(r) > U(s)$ yields $u(a_2) > .75u(a_1) + .25u(a_3)$. Such $u(a_2)$ will always exist because we know that $u(a_1) < u(a_2) < u(a_3)$. Let's do an example. Set $u(a_1) = -5$ and $u(a_3) = 5$. We can now pick any $u(a_2) \in (-2.5, 5)$, so let's try $u(a_2) = -1$.

$$U(p) = 2 > U(q) = 1.6 > U(r) = 0.8 > U(s) = 0.5.$$

We conclude that the information is consistent with the preferences being representable by the expected utility of a payoff function.

We now check the second piece of information. If the preferences can be represented by the expected value of a utility function, it must be the case that

$$\begin{aligned} .4u(a_1) + .6u(a_3) &> .5u(a_2) + .5u(a_3) \\ &> .3u(a_1) + .2u(a_2) + .5u(a_3) > .45u(a_1) + .55u(a_3) \end{aligned}$$

We solve the first and the last inequalities for $u(a_2)$ and obtain

$$u(a_2) < .8u(a_1) + .2u(a_3) \quad (1)$$

$$u(a_2) > .75u(a_1) + .25u(a_3) \quad (2)$$

This implies $.8u(a_1) + .2u(a_3) > .75u(a_1) + .25u(a_3)$. However, because $u(a_1) < u(a_3)$, it follows that $.8u(a_1) + .2u(a_3) < .75u(a_1) + .25u(a_3)$, a contradiction. Therefore, there is no $u(a_2)$ that can simultaneously solve (1) and (2). Therefore, the second piece of information is not consistent with the preferences being representable by the expected value of a payoff function.

QUESTION 4. (NORMALIZED BERNOULLI PAYOFF FUNCTIONS.) Suppose that a decision-maker's preferences can be represented by the expected value of a Bernoulli payoff function $u(\cdot)$. Find a Bernoulli payoff function whose expected value also represents the decision-maker's preferences and that assigns a payoff of 1 to the best outcome and the payoff 0 to the worst outcome.

Answer. We make use of Proposition 5 from the lecture notes: If U is a vNM expected utility function, then $\hat{U}$ is another vNM expected utility function iff there exist scalars $a > 0$ and b , such that $\hat{U}(p) = aU(p) + b$. Denote the best outcome by $\bar{x}$ and the worst outcome by $\underline{x}$. We know that $U(p) = \sum_x p(x)u(x)$, where $u(\cdot)$ is the Bernoulli payoff function. Because $U(\cdot)$ is linear, all we have to do is find a new Bernoulli payoff function, $v(x) = au(x) + b$, with the provision that $v(\bar{x}) = 1$ and $v(\underline{x}) = 0$. When $v(\cdot)$ is a linear transformation of $u(\cdot)$, we know that its expected value is also going to represent the preferences of the decision-maker. Thus, we solve the following system of equations:

$$au(\underline{x}) + b = 0$$

$$au(\bar{x}) + b = 1.$$

The solutions are

$$a = \frac{1}{u(\bar{x}) - u(\underline{x})} > 0 \quad b = \frac{-u(\underline{x})}{u(\bar{x}) - u(\underline{x})},$$

where the inequality follows from the fact that $u(\bar{x}) > u(\underline{x})$. Therefore, the equivalent payoff function is

$$v(x) = \frac{u(x) - u(\underline{x})}{u(\bar{x}) - u(\underline{x})}.$$

This is called “normalization” of the utility function and can be very useful for simplifying the payoff structures of games.