

LOGIC & PROBABILITY THEORY

Show all steps and calculations in your answers, for proofs justify each step.

QUESTION 1. Assume (a) $s \Rightarrow r$, (b) $[p \vee q] \Rightarrow \neg r$, (c) $\neg s \Rightarrow [\neg q \Rightarrow r]$, (d) p . Prove q .

Answer. Straightforward proof:

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|----|------------------------|----------------------------|
| 1. | $p \vee q$ | (d), addition |
| 2. | $\neg r$ | (1) and (b), modus ponens |
| 3. | $\neg s$ | (2) and (a), modus tollens |
| 4. | $\neg q \Rightarrow r$ | (3) and (c), modus ponens |
| 5. | $\neg \neg q$ | (4) and (2), modus tollens |
| 6. | q | (5), double negation |

QUESTION 2. Assume (a) $q \Rightarrow p$, (b) $\neg[\neg q \vee s]$, (c) $p \Rightarrow s$. Prove q . Prove $\neg q$. Comment.

Answer. First we prove q , and then we prove $\neg q$:

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|----|-----------------------------|----------------------|----|-----------------------------|----------------------------|
| 1. | $\neg \neg q \wedge \neg s$ | (b), DeMorgan | 1. | $\neg \neg q \wedge \neg s$ | (b), DeMorgan |
| 2. | $\neg \neg q$ | (1), simplification | 2. | $\neg s$ | (1), simplification |
| 3. | q | (2), double negation | 3. | $\neg p$ | (2) and (c), modus tollens |
| | | | 4. | $\neg q$ | (3) and (a), modus tollens |

The premises are contradictory because they claim first that $q \Rightarrow s$ [from (a) and (c) by the hypothetical syllogism], and that q and $\neg s$ are both true [from (b) by DeMorgan]. But the latter cannot be the case if q implies s . The proof demonstrates that if you allow some contradiction in your argument, then *anything* can be proven true. It also shows that sometimes the contradiction may not be evident when one just lists assumptions.

QUESTION 3. If the moon is made of cheese, then pigs can fly or game theory is fun. Pigs cannot fly. Game theory is fun. Therefore, the moon is not made of cheese. Write this in symbolic form and show whether the argument is valid or not. Comment.

Answer. This is mildly ridiculous, but here it goes. Let p denote the statement "the moon is made of cheese," q denote the statement "pigs can fly," and r denote the statement "game theory is fun." Rewrite the claim as follows: Assume (a) $p \Rightarrow [q \vee r]$, (b) $\neg q$, (c) r . Then $\neg p$. The argument is invalid. By (b) and (c), we know that $[q \vee r] \equiv s$ is true. Unfortunately this tells us nothing about p because from $p \Rightarrow s$ and s one cannot infer anything about p . To see this, let's come up with an example that satisfies the assumptions when p is either true or false. Suppose p is true. Since $p \Rightarrow s$, we now infer that s is true, which accords with the assumption. Suppose now that p is false. Since s is false, $p \Rightarrow s$ is true (consult the truth-table if you can't remember that). In other words, this also satisfies both assumptions. Hence, the way the argument is stated cannot lead to the conclusion that the moon is not made of cheese. This shows that you can have valid conclusions with invalid arguments, so beware!

QUESTION 4. Prove formally by contradiction Aristotle's famous syllogism: All men are mortal. Socrates is a man. Therefore, Socrates is mortal.

Answer. Let $p(x)$ denote the property “ x is a man” and $q(x)$ denote the property “ x is mortal”. Then, the statement “all men are mortal” can be written as $\forall x[p(x) \Rightarrow q(x)]$. Let s denote Socrates, and so the statement “Socrates is a man” can be written as $p(s)$. The syllogism can be rewritten as follows. Assume (a) $\forall x[p(x) \Rightarrow q(x)]$, and (b) $p(s)$. Prove $q(s)$. Let’s do a proof by contradiction, just for fun:

1. $\neg q(s)$ assumed premise (Socrates is immortal)
2. $p(s) \Rightarrow q(s)$ (a), universal instantiation
3. $\neg p(s)$ (1) and (2), modus tollens (Socrates is not a man)
4. $p(s) \wedge \neg p(s)$ (3) and (b), conjunction and contradiction

Of course, you could just do a straight proof by applying universal instantiation to (a) and then using modus ponens on the result with (b). But then you won’t reach the conclusion that Socrates is not a man.

QUESTION 5. Write the following statements symbolically: (a) Every person is better at something than someone else; (b) In anything any person does, someone else is better than that person. *Hint: use a 2-place predicate of the form $q(x, y)$ to express relations between x and y .*

Answer. Rephrasing in more convenient ways helps:

- (a) For every person x , there is a person y such that x is better at something than y . We have to use both the universal and existential quantifiers here. Let $p(x)$ be the predicate “ x is a person” and $q(x, y)$ be the relation “ x is better at something than y ”. The statement then is:

$$\forall x \left[p(x) \Rightarrow \left(\exists y [p(y) \wedge q(x, y)] \right) \right].$$

- (b) For every task z and every person x that does z , there is a person y that does z better than x . Let $s(x)$ denote the predicate “ x is a task,” $p(x)$ denote “ x is a person,” $r(x, z)$ denote “ x does z , and $q(x, y, z)$ denote the relation “ y is better than x at z ”. The statement then is:

$$\forall z \left[s(z) \Rightarrow \forall x \left[p(x) \wedge r(x, z) \Rightarrow \exists y [p(y) \wedge r(y, z) \wedge q(x, y, z)] \right] \right].$$

QUESTION 6. If $\Pr(AB) > \Pr(A)\Pr(B)$, events A and B are positively correlated. Consider an experiment that consists of rolling a fair die twice. Let X_1 and X_2 be the scores on the first and second roll, respectively, and let $Y = X_1 + X_2$. Using this experiment, show that correlation is not transitive.

Answer. We want to show that there exist numbers a, b , and y such that even though $X_1 = a$ and $Y = y$ are positively correlated, and $Y = y$ and $X_2 = b$ are positively correlated, $X_1 = a$ and $X_2 = b$ are not correlated. Take $a = 3$, $b = 2$, and $y = 5$. Because positive correlation can also be expressed as $\Pr(A|B) > \Pr(A)$, we can calculate the probabilities as follows:

$$\begin{aligned} \Pr(X_1 = 3|Y = 5) &= \frac{\Pr(Y = 5|X_1 = 3) \Pr(X_1 = 3)}{\Pr(Y = 5)} = \frac{(1/6)(1/6)}{1/9} = 1/4 > 1/6 = \Pr(X_1 = 3) \\ \Pr(X_2 = 2|Y = 5) &= \frac{\Pr(Y = 5|X_2 = 2) \Pr(X_2 = 2)}{\Pr(Y = 5)} = \frac{(1/6)(1/6)}{1/9} = 1/4 > 1/6 = \Pr(X_2 = 2). \end{aligned}$$

Here we use the fact that the probability of obtaining any particular Y conditional on the realization of the first die roll equals the probability that the second roll yields the unique difference $X_2 = Y - X_1$, which is just $1/6$ if $X_2 \leq 6$ (so it’s feasible) and 0 otherwise. For example, $\Pr(Y = 12|X_1 = 1) = 0$ because the maximum value Y can obtain now is just 7. We further noted that there are four ways $Y = 5$ out of 36

possible realizations, and so $\Pr(Y = 5) = 4/36 = 1/9$. We now have shown that X_1 and Y , as well as Y and X_2 are positively correlated. We now want to show that X_1 and X_2 are not:

$$\Pr(X_1 = 3|X_2 = 2) = \frac{\Pr(X_2 = 2|X_1 = 3) \Pr(X_1 = 3)}{\Pr(X_2 = 2)} = \frac{(1/6)(1/6)}{1/6} = 1/6 = \Pr(X_1 = 3),$$

which we already knew since the two rolls are independent. (You could also show that even if both $X_1 = a$ and $Y = y$, and $X_1 = b$ and $Y = y$ are positively correlated, $X_1 = a$ and $X_1 = b$ are negatively correlated. The last part is easy: if the first roll results in some a , then the probability it will yield $b \neq a$ conditional on having yielded a is $0 < \Pr(X_1 = b)$.)

QUESTION 7. A fair die is rolled, and then a fair coin is tossed the number of times showing on the die. What is the probability that all the coin tosses show heads? Given that all coins are heads, what is the probability that the die score was 4?

Answer. Let n be the number showing on the die, then $\Pr(\text{all heads}|n) = (1/2)^n$. Since the probability of any $n \leq 6$ is $1/6$, and the realizations are exhaustive and mutually exclusive, we have:

$$\Pr(\text{all heads}) = \frac{1}{6} \cdot \sum_{n=1}^6 (1/2)^n = \left(\frac{1}{6}\right) \left(\frac{63}{64}\right) = \frac{21}{128}.$$

To answer the second question, we write out the conditional probability:

$$\Pr(n = 4|\text{all heads}) = \frac{\Pr(\text{all heads}|n = 4) \Pr(n = 4)}{\Pr(\text{all heads})} = \frac{(1/2)^4(1/6)}{21/128} = \left(\frac{1}{96}\right) \left(\frac{128}{21}\right) = \frac{4}{63}.$$

QUESTION 8. Suppose that in a population in which the incidence of thyroid disorder is 5%, 10% of women, and 1% of men have that disorder. A randomly chosen person is found to have thyroid disorder. What is the probability that this person is female?

Answer. Let M be the event “person is male” and F denote the event “person is female.” Let T denote the event “has thyroid disorder.” We know that the probability of a man having the disorder is $\Pr(T|M) = 0.01$, and the probability of a woman having the disorder is $\Pr(T|F) = 0.10$. We want to know $\Pr(F|T)$. From Bayes rule:

$$\Pr(F|T) = \frac{\Pr(T|F) \Pr(F)}{\Pr(T|F) \Pr(F) + \Pr(T|M) \Pr(M)}.$$

We now need to find $\Pr(F)$ and $\Pr(M)$. Let f and m denote the total numbers of women and men, respectively, and so $f + m$ is the size of the population. Since the incidence of thyroid is 5%, and the sick people are either men or women, we have

$$.05(f + m) = .10f + .01m \Leftrightarrow f = 4/5 \cdot m,$$

that is, the number of women is only $4/5$ the number of men. The probability that a randomly chosen person is female, then, is:

$$\Pr(F) = \frac{f}{f + m} = \frac{4/5 \cdot m}{4/5 \cdot m + m} = 4/9,$$

which means that $\Pr(M) = 1 - \Pr(F) = 5/9$. Putting everything together, we obtain:

$$\Pr(F|T) = \frac{1/10 \cdot 4/9}{1/10 \cdot 4/9 + 1/100 \cdot 5/9} = 40/45.$$

The probability that a randomly chosen person found to have thyroid disorder is female is about 89%.

QUESTION 9. Two fair dice are rolled and the sequence of scores (X_1, X_2) is recorded. Find the probability function of the random variable $Y = \min\{X_1, X_2\}$.

Answer. For any $Y = n$, three configurations are possible: $(X_1 = n, X_2 > n)$, $(X_1 > n, X_2 = n)$, or $(X_1 = n, X_2 = n)$. Taking the first configuration with $X_1 = n$, there are $6 - n$ ways to get $X_2 > n$. Analogously, taking the second with $X_2 = n$, there are $6 - n$ ways to get $X_1 > n$. There is only one $(X_1 = n, X_2 = n)$ for any n . Therefore, the total number of ways in which we can obtain $Y = n$ is $2(6 - n) + 1$. Because each unique way has a probability of $1/36$, the probability function is:

$$f(n) = 1/36 \times [2(6 - n) + 1].$$

Let's check:

$$\begin{array}{lll} f(1) = 11/36 & f(2) = 9/36 & f(3) = 7/36 \\ f(4) = 5/36 & f(5) = 3/36 & f(6) = 1/36. \end{array}$$

Since these are all the possible realizations, we must get $\sum_n f(n) = 1$, which is indeed the case.

QUESTION 10. An urn contains 25 green and 40 red balls. A sample of 5 balls is selected at random. Let Y denote the number of red balls in the sample. What is the probability function of Y ? What is $\Pr(Y > 3)$?

Answer. Because the ordering is irrelevant, in this sampling without replacement, the sample space consists of combinations of length $k = 5$. We have $n = 65$ balls, and so a total of $\binom{65}{5} = 8,259,888$ possible samples, each of which is equally likely. There are 25 green balls, so there are $\binom{25}{5} = 53,130$ samples that contain exactly five green balls (no red ones). Analogously, there are $\binom{40}{5} = 658,008$ samples that contain exactly five red balls (no green ones). To compute the number of samples that contain exactly one red ball, note that there are 40 ways to get one red ball, and $\binom{25}{4}$ ways to match it with 4 green ones. In other words, there are $40 \times \binom{25}{4} = 40 \times 12,650 = 506,000$ samples with one red ball. There are $\binom{40}{2} = 780$ ways to obtain two red balls, and $\binom{25}{3}$ ways to obtain 3 green ones, so the total number of samples with 2 red and 3 green balls is $\binom{40}{2} \times \binom{25}{3} = 780 \times 2,300 = 1,794,000$. There are $\binom{40}{3}$ ways to obtain three red balls, and $\binom{25}{2}$ ways to match them with 2 green ones, so the total number of samples with 3 red and 2 green balls is $\binom{40}{3} \times \binom{25}{2} = 9,880 \times 300 = 2,964,000$. Finally, there are $\binom{40}{4}$ ways to obtain 4 red balls, and 25 ways to match them with a green ball, so the total number of samples with 4 red and 1 green balls is $\binom{40}{4} \times 25 = 91,390 \times 25 = 2,284,750$. Let's check our math: adding the number of samples yields a total of 8,259,888 samples, which is what we obtained with our formula above. The pattern is obvious:

$$f(0) = \frac{\binom{40}{0}\binom{25}{5}}{\binom{65}{5}} \quad f(1) = \frac{\binom{40}{1}\binom{25}{4}}{\binom{65}{5}} \quad \dots \quad f(4) = \frac{\binom{40}{4}\binom{25}{1}}{\binom{65}{5}} \quad f(5) = \frac{\binom{40}{5}\binom{25}{0}}{\binom{65}{5}}.$$

More compactly, the distribution formula for Y red balls in a random sample (without replacement) of size k from an urn with n balls of which r are red and $n - r$ are green, is:

$$f(Y) = \frac{\binom{r}{Y}\binom{n-r}{k-Y}}{\binom{n}{k}},$$

which, of course, is the hypergeometric distribution. The second question is really easy now:

$$\Pr(Y > 3) = \Pr(Y = 4) + \Pr(Y = 5) = f(4) + f(5).$$

I will leave it to you to calculate this.