

# Game Theory: Dominance, Nash Equilibrium, Symmetry

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# 1 Elimination of Dominated Strategies

## 1.1 Strict Dominance in Pure Strategies

In some games, a player's strategy is superior to all other strategies *regardless of what the other players do*. This strategy then *strictly dominates* the other strategies. Consider the Prisoner's Dilemma game in Fig. 1 (p. 2). Choosing *D* strictly dominates choosing *C* because it yields a better payoff regardless of what the other player chooses to do.

If one player is going to play *D*, then the other is better off by playing *D* as well. Also, if one player is going to play *C*, then the other is better off by playing *D* again. For each prisoner, choosing *D* is always better than *C* regardless of what the other prisoner does. We say that *D* strictly dominates *C*.

|            |          |            |          |
|------------|----------|------------|----------|
|            |          | Prisoner 2 |          |
|            |          | <i>C</i>   | <i>D</i> |
| Prisoner 1 | <i>C</i> | 2,2        | 0,3      |
|            | <i>D</i> | 3,0        | 1,1      |

Figure 1: Prisoner's Dilemma.

DEFINITION 1. In the strategic form game  $G$ , let  $s'_i, s''_i \in S_i$  be two strategies for player  $i$ . Strategy  $s'_i$  **strictly dominates** strategy  $s''_i$  if

$$u_i(s'_i, s_{-i}) > u_i(s''_i, s_{-i})$$

for every strategy profile  $s_{-i} \in S_{-i}$ .

In words, a strategy  $s'_i$  strictly dominates  $s''_i$  if for each feasible combination of the other players' strategies,  $i$ 's payoff from playing  $s'_i$  is strictly greater than the payoff from playing  $s''_i$ . Also, strategy  $s''_i$  is **strictly dominated** by  $s'_i$ . In the PD game, *Defect* strictly dominates *Cooperate*, and *Cooperate* is strictly dominated by *Defect*.

Rational players never play strictly dominated strategies, because such strategies can never be best responses to any strategies of the other players. There is no belief that a rational player can have about the behavior of other players such that it would be optimal to choose a strictly dominated strategy. Thus, in PD a rational player would never choose *C*. We can use this concept to find solutions to some simple games. For example, since neither player will ever choose *C* in PD, we can **eliminate** this strategy from the strategy space, which means that now both players only have one strategy left to them: *D*. The solution is now trivial: It follows that the only possible rational outcome is  $\langle D, D \rangle$ .

Because players would never choose strictly dominated strategies, eliminating them from consideration should not affect the analysis of the game because this fact should be evident to all players in the game. In the PD example, eliminating strictly dominated strategies resulted in a unique prediction for how the game is going to be played. The concept is more general, however, because even in games with more strategies, eliminating a strictly dominated one may result in other strategies becoming strictly dominated in the game that remains.

Consider the abstract game depicted in Fig. 2 (p. 3). Player 1 does not have a strategy that is strictly dominated by another: playing *U* is better than *M* unless player 2 chooses *C*, in which case *M* is better. Playing *D* is better than playing *U* unless player 2 chooses *R*, in which

case  $U$  is better. Finally, playing  $D$  instead of  $M$  is better unless player 2 chooses  $R$ , in which case  $M$  is better.

|          |     | Player 2 |     |     |
|----------|-----|----------|-----|-----|
|          |     | $L$      | $C$ | $R$ |
| Player 1 | $U$ | 4,3      | 5,1 | 6,2 |
|          | $M$ | 2,1      | 8,4 | 3,6 |
|          | $D$ | 5,9      | 9,6 | 2,8 |

Figure 2: A  $3 \times 3$  Example Game.

For player 2, on the other hand, strategy  $C$  is strictly dominated by strategy  $R$ . Notice that whatever player 1 chooses, player 2 is better off playing  $R$  than playing  $C$ : she gets  $2 > 1$  if player 1 chooses  $U$ ; she gets  $6 > 4$  if player 1 chooses  $M$ ; and she gets  $8 > 6$  if player 1 chooses  $D$ . Thus, a rational player 2 would never choose to play  $C$  when  $R$  is available. (Note here that  $R$  neither dominates, nor is dominated by,  $L$ .) If player 1 knows that player 2 is rational, then player 1 would play the game *as if* it were the game depicted in Fig. 3 (p. 3).

|          |     | Player 2 |     |
|----------|-----|----------|-----|
|          |     | $L$      | $R$ |
| Player 1 | $U$ | 4,3      | 6,2 |
|          | $M$ | 2,1      | 3,6 |
|          | $D$ | 5,9      | 2,8 |

Figure 3: The Reduced Example Game, Step I.

We examine player 1's strategies again. We now see that  $U$  strictly dominates  $M$  because player 1 gets  $4 > 2$  if player 2 chooses  $L$ , and  $6 > 3$  if player 2 chooses  $R$ . Thus, a rational player 1 would never choose  $M$  given that he knows player 2 is rational as well and consequently will never play  $C$ . (Note that  $U$  neither dominates, nor is dominated by,  $D$ .)

If player 2 knows that player 1 is rational and knows that player 1 knows that she is also rational, then player 2 would play the game *as if* it were the game depicted in Fig. 4 (p. 3).

|          |     | Player 2 |     |
|----------|-----|----------|-----|
|          |     | $L$      | $R$ |
| Player 1 | $U$ | 4,3      | 6,2 |
|          | $D$ | 5,9      | 2,8 |

Figure 4: The Reduced Example Game, Step II.

We examine player 2's strategies again and notice that  $L$  now strictly dominates  $R$  because player 2 would get  $3 > 2$  if player 1 chooses  $U$ , and  $9 > 8$  if player 1 chooses  $D$ . Thus, a rational player 2 would never choose  $R$  given that she knows that player 1 is rational, etc.

If player 1 knows that player 2 is rational, etc., then he would play the game *as if* it were the game depicted in Fig. 5 (p. 4).

But now,  $U$  is strictly dominated by  $D$ , so player 1 would never play  $U$ . Therefore, player 1's rational choice here is to play  $D$ . This means the outcome of this game will be  $\langle D, L \rangle$ , which yields player 1 a payoff of 5 and player 2 a payoff of 9.

The process described above is called **iterated elimination of strictly dominated strategies**. The solution of  $G$  is the equilibrium  $\langle D, L \rangle$ , and is sometimes called **iterated-dominance**.

|          |     | Player 2                                                             |  |     |     |
|----------|-----|----------------------------------------------------------------------|--|-----|-----|
|          |     | L                                                                    |  |     |     |
| Player 1 | U   | <table border="1"><tr><td>4,3</td></tr><tr><td>5,9</td></tr></table> |  | 4,3 | 5,9 |
|          | 4,3 |                                                                      |  |     |     |
| 5,9      |     |                                                                      |  |     |     |
| D        |     |                                                                      |  |     |     |

Figure 5: The Reduced Example Game, Step III.

**equilibrium**, or *iterated-dominant strategy equilibrium*. The game  $G$  is sometimes called **dominance-solvable**.

Although the process is intuitively appealing (after all, rational players would never play strictly dominated strategies), each step of elimination requires a further assumption about the other player's rationality. Recall that we started by assuming that player 1 knows that player 2 is rational and so she would not play  $C$ . This allowed the elimination of  $M$ . Next, we had to assume that player 2 knows that player 1 is rational *and* that she also knows that player 1 knows that she herself is rational as well. This allowed the elimination of  $R$ . Finally, we had to assume that player 1 knows that player 2 is rational and that he also knows that player 2 knows that player 1 is rational and that player 2 also knows that player 1 knows that player 2 is rational.

More generally, we want to be able to make this assumption for as many iterations as might be needed. That is, we must be able to assume not only that all players are rational, but also that all players know that all the players are rational, and that all the players know that all the players know that all players are rational, and so on, *ad infinitum*. This assumption is called **common knowledge** and is usually made in game theory.

## 1.2 Weak Dominance

Rational players would never play strictly dominated strategies, and so eliminating these should not affect our analysis. There may be circumstances, however, where a strategy is “not worse” than another instead of being “always better” (as a strictly dominant one would be). To define this concept, we introduce the idea of **weakly dominated strategy**.

DEFINITION 2. In the strategic form game  $G$ , let  $s'_i, s''_i \in S_i$  be two strategies for player  $i$ . Strategy  $s'_i$  **weakly dominates** strategy  $s''_i$  if

$$u_i(s'_i, s_{-i}) \geq u_i(s''_i, s_{-i})$$

for every strategy profile  $s_{-i} \in S_{-i}$ , and there exists at least one  $s_{-i}$  such that the inequality is strict.

In other words,  $s'_i$  never does worse than  $s''_i$ , and sometimes does better. While iterated elimination of strictly dominated strategies seems to rest on rather firm foundation (except for the common knowledge requirement that might be a problem with more complicated situations), eliminating weakly dominated strategies is more controversial because it is harder to argue that it should not affect analysis. The reason is that by definition, a weakly dominated strategy can be a best response for the player. Furthermore, there are technical difficulties with eliminating weakly dominated strategies: the order of elimination can matter for the result!

Consider the game in Fig. 6 (p. 5). Strategy  $D$  is strictly dominated by  $U$ , so if we remove it first, we are left with a game, in which  $L$  weakly dominates  $R$ . Eliminating  $R$  in turn results

|          |          | Player 2 |          |
|----------|----------|----------|----------|
|          |          | <i>L</i> | <i>R</i> |
| Player 1 | <i>U</i> | 3, 2     | 2, 2     |
|          | <i>M</i> | 1, 1     | 0, 0     |
|          | <i>D</i> | 0, 0     | 1, 1     |

Figure 6: The Kohlberg and Mertens Game.

in a game where  $U$  strictly dominates  $M$ , and so the prediction is  $\langle U, L \rangle$ . However, note that  $M$  is strictly dominated by  $U$  in the original game as well. If we begin by eliminating  $M$ , then  $R$  weakly dominates  $L$  in the resulting game. Eliminating  $L$  in turn results in a game where  $U$  strictly dominates  $D$ , and so the prediction is  $\langle U, R \rangle$ . If we begin by eliminating  $M$  and  $D$  at the same time, then we are left with a game where neither of the strategies for player 2 weakly dominates the other. Thus, the order in which we eliminate the strictly dominated strategies for player 1 determines which of player 2's weakly dominated strategies will get eliminated in the iterative process.

This dependence on the order of elimination does not arise if we only eliminated strictly dominated strategies. If we perform the iterative process until no strictly dominated strategies remain, the resulting game will be the same regardless of the order in which we perform the elimination. Eliminating strategies for other players can never cause a strictly dominated strategy to cease to be dominated but it can cause a weakly dominated strategy to cease being dominated. Intuitively, you should see why the latter might be the case. For a strategy  $s_i$  to be weakly dominated, all that is required is that some other strategy  $s'_i$  is as good as  $s_i$  for all strategies  $s_{-i}$  and only better than  $s_i$  for one strategy of the opponent. If that particular strategy gets eliminated, then  $s_i$  and  $s'_i$  yield the same payoffs for all remaining strategies of the opponent, and neither weakly dominates the other.

### 1.3 Strict Dominance and Mixed Strategies

We now generalize the idea of dominance to mixed strategies. All that we have to do to decide whether a pure strategy is dominated is to check whether there exists some mixed-strategy that is a better response to all pure strategies of the opponents.

**DEFINITION 3.** In a strategic form game  $G$  with vNM preferences, the pure strategy  $s_i$  is **strictly dominated for player  $i$**  if there exists a mixed strategy  $\sigma_i \in \Sigma_i$  such that

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \text{ for every } s_{-i} \in S_{-i}. \quad (1)$$

The strategy  $s_i$  is **weakly dominated** if there exists a  $\sigma_i$  such that inequality (1) holds with weak inequality, and the inequality is strict for at least one  $s_{-i}$ .

Note that when checking if a pure strategy is dominated by a mixed strategy, we only consider pure strategies for the rest of the players. This is because for a given  $s_i$ , the strategy  $\sigma_i$  satisfies (1) for all pure strategies of the opponents if, and only if, it satisfies it for all mixed strategies  $\sigma_{-i}$  as well because player  $i$ 's payoff when his opponents play mixed strategies is a convex combination of his payoffs when they play pure strategies.

As a first example of a pure strategy dominated by a mixed strategy, consider our favorite card game, whose strategic form, reproduced in Fig. 7 (p. 6), we have derived before. Consider

strategy  $s_1 = Ff$  for player 1 and the mixed strategy  $\sigma_1 = (0.5)[Rr] + (0.5)[Fr]$ . We now have:

$$u_1(\sigma_1, m) = (0.5)(0) + (0.5)(0.5) = 0.25 > 0 = u_1(s_1, m)$$

$$u_1(\sigma_1, p) = (0.5)(1) + (0.5)(0) = 0.5 > 0 = u_1(s_1, p).$$

In other words, playing  $\sigma_1$  yields a higher expected payoff than  $s_1$  does against any possible strategy for player 2. Therefore,  $s_1$  is strictly dominated by  $\sigma_1$ , and we should not expect player 1 to play  $s_1$ . On the other hand, the strategy  $Fr$  only weakly dominates  $Ff$  because it yields a strictly better payoff against  $m$  but the same payoff against  $p$ . Eliminating weakly dominated strategies is much more controversial than eliminating strictly dominated ones (we shall see why in the homework).

|          |      | Player 2  |       |
|----------|------|-----------|-------|
|          |      | $m$       | $p$   |
| Player 1 | $Rr$ | 0, 0      | 1, -1 |
|          | $Rf$ | -0.5, 0.5 | 1, -1 |
|          | $Fr$ | 0.5, -0.5 | 0, 0  |
|          | $Ff$ | 0, 0      | 0, 0  |

Figure 7: The Strategic Form of the Myerson Card Game.

As an example of iterated elimination of strictly dominated strategies that can involve mixed strategies, consider the game in Fig. 8 (p. 6). None of the pure strategies for player 1 strictly dominates any of his other strategies. Further, no mixed strategy for player 1 strictly dominates any of his pure strategies. This is easy to see: since  $U$  is a best response to  $C$ , it cannot be dominated by any mixed strategy that assigns positive probability to  $D$  because such a strategy would yield a strictly lower payoff against  $C$ . A similar argument establishes that  $D$  cannot be dominated by any mixed strategy because  $D$  is a best response to  $R$ .

|          |     | Player 2 |      |      |
|----------|-----|----------|------|------|
|          |     | $L$      | $C$  | $R$  |
| Player 1 | $U$ | 2, 3     | 3, 0 | 0, 1 |
|          | $D$ | 0, 0     | 1, 6 | 4, 2 |

Figure 8: Another Game from Myerson (p. 58).

Further, none of the pure strategies strictly dominates any other strategy for player 2. Let's see if we can find a mixed strategy for player 2 that would strictly dominate one of her pure strategies. Which pure strategy should we try to eliminate? It cannot be  $C$  because any mixture between  $L$  and  $R$  would involve a convex combination of 0 and 2 against  $D$  which can never exceed 6, which is what  $C$  would yield in this case. It cannot be  $L$  either because it yields 3 against  $U$ , and any mixture between  $C$  and  $R$  against  $U$  would yield at most 1. Hence, let's try to eliminate  $R$ : one can imagine mixtures between  $L$  and  $C$  that would yield a payoff higher than 1 against  $U$  and higher than 2 against  $D$ . One such mixture would be playing them both with equal probability. The mixed strategy  $\sigma_2 = (0.5)[L] + (0.5)[C] = (.5, .5, 0)$  strictly dominates the pure strategy  $R$ . To see this, note that

$$u_2(\sigma_2, U) = (0.5)(3) + (0.5)(0) = 1.5 > 1 = u_2(R, U)$$

$$u_2(\sigma_2, D) = (0.5)(0) + (0.5)(6) = 3 > 2 = u_2(R, D).$$

We can therefore eliminate  $R$ , which produces the game in Fig. 9 (p. 7).

|          |     | Player 2 |      |
|----------|-----|----------|------|
|          |     | $L$      | $C$  |
| Player 1 | $U$ | 2, 3     | 3, 0 |
|          | $D$ | 0, 0     | 1, 6 |

Figure 9: The Game from Fig. 8 (p. 6) after Elimination of  $R$ .

In this game, strategy  $U$  strictly dominates  $D$  because  $2 > 0$  against  $L$  and  $3 > 1$  against  $C$ . Therefore, because player 1 knows that player 2 is rational and would never choose  $R$ , he can eliminate  $D$  from his own choice set. But now player 2's choice is also simple because in the resulting game  $L$  strictly dominates  $C$  because  $3 > 0$ . She therefore eliminates this strategy from her choice set. The iterated elimination of strictly dominated strategies leads to a unique prediction as to what rational players should do in this game:  $\langle U, L \rangle$ .

The following several remarks are useful observations about the relationship between dominance and mixed strategies. Each is easily verifiable by example.

REMARK 1. A mixed strategy that assigns positive probability to a dominated pure strategy is itself dominated (by any other mixed strategy that assigns less probability to the dominated pure strategy).

REMARK 2. A mixed strategy may be strictly dominated even though it assigns positive probability only to pure strategies that are not even weakly dominated.

|          |     | Player 2 |       |
|----------|-----|----------|-------|
|          |     | $L$      | $R$   |
| Player 1 | $U$ | 1, 3     | -2, 0 |
|          | $M$ | -2, 0    | 1, 3  |
|          | $D$ | 0, 1     | 0, 1  |

Figure 10: Mixed Strategy Dominated by a Pure Strategy.

Consider the example in Fig. 10 (p. 7). Playing  $U$  and  $M$  with probability  $1/2$  each gives player 1 an expected payoff of  $-1/2$  regardless of what player 2 does. This is strictly dominated by the pure strategy  $D$ , which gives him a payoff of 0, even though neither  $U$  nor  $M$  is (even weakly) dominated.

REMARK 3. A strategy not strictly dominated by any other pure strategy may be strictly dominated by a mixed strategy.

|          |     | Player 2 |      |
|----------|-----|----------|------|
|          |     | $L$      | $R$  |
| Player 1 | $U$ | 1, 3     | 1, 0 |
|          | $M$ | 4, 0     | 0, 3 |
|          | $D$ | 0, 1     | 3, 1 |

Figure 11: Pure Strategy Dominated by a Mixed Strategy.

Consider the example in Fig. 11 (p. 7). Playing  $U$  is not strictly dominated by either  $M$  or  $D$  and gives player 1 a payoff of 1 regardless of what player 2 does. This is strictly dominated by the mixed strategy in which player 1 chooses  $M$  and  $D$  with probability  $1/2$  each, which would yield 2 if player 2 chooses  $L$  and  $3/2$  if player 2 chooses  $R$ , and so it would yield at least  $3/2 > 1$  regardless of what player 2 does.

The iterated elimination of strictly dominated strategies is quite intuitive but it has a very important drawback. Even though the dominant strategy equilibrium is unique if it exists, for most games that we wish to analyze, all strategies (or too many of them) will survive iterated elimination, and there will be no such equilibrium. Thus, this solution concept will leave many games “unsolvable” in the sense that we shall not be able predict how rational players will play them. In contrast, the concept of Nash equilibrium, to which we turn now, has the advantage that it exists in a very broad class of games.

## 2 Nash Equilibrium

### 2.1 Pure-Strategy Nash Equilibrium

Rational players think about actions that the other players might take. In other words, players form beliefs about one another’s behavior. For example, in the BoS game, if the man believed the woman would go to the ballet, it would be prudent for him to go to the ballet as well. Conversely, if he believed that the woman would go to the fight, it is probably best if he went to the fight as well. So, to maximize his payoff, he would select the strategy that yields the greatest expected payoff given his belief. Such a strategy is called a **best response** (or *best reply*).

DEFINITION 4. Suppose player  $i$  has some belief  $s_{-i} \in S_{-i}$  about the strategies played by the other players. Player  $i$ ’s strategy  $s_i \in S_i$  is a **best response** if

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \text{ for every } s'_i \in S_i.$$

We now define the **best response correspondence**,  $BR_i(s_{-i})$ , as the set of best responses player  $i$  has to  $s_{-i}$ . It is important to note that the best response correspondence is set-valued. That is, there may be more than one best response for any given belief of player  $i$ . If the other players stick to  $s_{-i}$ , then player  $i$  can do no better than using any of the strategies in the set  $BR_i(s_{-i})$ . In the BoS game, the set consists of a single member:  $BR_m(F) = \{F\}$  and  $BR_m(B) = \{B\}$ . Thus, here the players have a single optimal strategy for every belief. In other games, like the one in Fig. 12 (p. 8),  $BR_i(s_{-i})$  can contain more than one strategy.

|          |     | Player 2 |     |     |
|----------|-----|----------|-----|-----|
|          |     | $L$      | $C$ | $R$ |
| Player 1 | $U$ | 2,2      | 1,4 | 4,4 |
|          | $M$ | 3,3      | 1,0 | 1,5 |
|          | $D$ | 1,1      | 0,5 | 2,3 |

Figure 12: The Best Response Game.

In this game,  $BR_1(L) = \{M\}$ ,  $BR_1(C) = \{U, M\}$ , and  $BR_1(R) = \{U\}$ . Also,  $BR_2(U) = \{C, R\}$ ,  $BR_2(M) = \{R\}$ , and  $BR_2(D) = \{C\}$ . You should get used to thinking of the best response correspondence as a set of strategies, one for each combination of the other players’ strategies.

(This is why we enclose the values of the correspondence in braces even when there is only one element.)

We can now use the concept of best responses to define Nash equilibrium: a Nash equilibrium is a strategy profile such that each player's strategy is a best response to the other players' strategies:

**DEFINITION 5 (Nash Equilibrium).** The strategy profile  $(s_i^*, s_{-i}^*) \in S$  is a **pure-strategy Nash equilibrium** if, and only if,  $s_i^* \in BR_i(s_{-i}^*)$  for each player  $i \in \mathcal{I}$ .

An equivalent useful way of defining Nash equilibrium is in terms of the payoffs players receive from various strategy profiles.

**DEFINITION 6.** The strategy profile  $(s_i^*, s_{-i}^*)$  is a **pure-strategy Nash equilibrium** if, and only if,  $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$  for each player  $i \in \mathcal{I}$  and each  $s_i \in S_i$ .

That is, for every player  $i$  and every strategy  $s_i$  of that player,  $(s_i^*, s_{-i}^*)$  is at least as good as the profile  $(s_i, s_{-i}^*)$  in which player  $i$  chooses  $s_i$  and every other player chooses  $s_{-i}^*$ . In a Nash equilibrium, no player  $i$  has an incentive to choose a different strategy when everyone else plays the strategies prescribed by the equilibrium. It is quite important to understand that a **strategy profile is a Nash equilibrium if no player has incentive to deviate from his strategy given that the other players do not deviate**. When examining a strategy for a candidate to be part of a Nash equilibrium (strategy profile), we always hold the strategies of all other players constant.<sup>1</sup>

To understand the definition of Nash equilibrium a little better, suppose there is some player  $i$ , for whom  $s_i$  is not a best response to  $s_{-i}$ . Then, there exists some  $s_i'$  such that  $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ . Then this (at least one) player has an incentive to deviate from the theory's prediction and these strategies are not Nash equilibrium.

Another important thing to keep in mind: Nash equilibrium is a strategy profile. Finding a solution to a game involves finding strategy profiles that meet certain rationality requirements. In strict dominance we required that neither of the player's strategies in equilibrium is strictly dominated. In Nash equilibrium, we require that each player's strategy is a best response to the strategies of the other players.

**THE PRISONERS' DILEMMA.** By examining all four possible strategy profiles, we see that  $(D, D)$  is the unique Nash equilibrium (NE). It is NE because (a) given that player 2 chooses  $D$ , then player 1 can do no better than chose  $D$  himself ( $1 > 0$ ); and (b) given that player 1 chooses  $D$ , player 2 can do no better than choose  $D$  himself. No other strategy profile is NE:

- $(C, C)$  is not NE because if player 2 chooses  $C$ , then player 1 can profitably deviate by choosing  $D$  ( $3 > 2$ ). Although this is enough to establish the claim, also note that the profile is not NE for another sufficient reason: if player 1 chooses  $C$ , then player 2 can profitably deviate by playing  $D$  instead. (Note that it is enough to show that one player can deviate profitably for a profile to be eliminated.)
- $(C, D)$  is not NE because if player 2 chooses  $D$ , then player 1 can get a better payoff by choosing  $D$  as well.

---

<sup>1</sup>There are several ways to motivate Nash equilibrium. Osborne offers the idea of social convention and Gibbons justifies it on the basis of self-enforcing predictions. Each has its merits and there are others (e.g. steady state in an evolutionary game). You should become familiar with these.

- $(D, C)$  is not NE because if player 1 chooses  $D$ , then player 2 can get a better payoff by choosing  $D$  as well.

Since this exhausts all possible strategy profiles,  $(D, D)$  is the unique Nash equilibrium of the game. It is no coincidence that the Nash equilibrium is the same as the strict dominance equilibrium we found before. In fact, as you will have to prove in your homework, a player will never use a strictly dominated strategy in a Nash equilibrium. Further, if a game is dominance solvable, then its solution is the unique Nash equilibrium.

How do we use best responses to find Nash equilibria? We proceed in two steps: First, we determine the best responses of each player, and second, we find the strategy profiles where strategies are best responses to each other.

For example, consider again the game in Fig. 12 (p. 8). We have already determined the best responses for both players, so we only need to find the profiles where each is best response to the other. An easy way to do this in the bi-matrix is by going through the list of best responses and marking the payoffs with a '\*' for the relevant player where a profile involves a best response. Thus, we mark player 1's payoffs in  $(U, C)$ ,  $(U, R)$ ,  $(M, L)$ , and  $(M, C)$ . We also mark player 2's payoffs in  $(U, C)$ ,  $(U, R)$ ,  $(M, R)$ , and  $(D, C)$ . This yields the matrix in Fig. 13 (p. 10).

|          |     | Player 2 |       |       |
|----------|-----|----------|-------|-------|
|          |     | $L$      | $C$   | $R$   |
| Player 1 | $U$ | 2,2      | 1*,4* | 4*,4* |
|          | $M$ | 3*,3     | 1*,0  | 1,5*  |
|          | $D$ | 1,1      | 0,5*  | 2,3   |

Figure 13: The Best Response Game Marked.

There are two profiles with stars for both players,  $(U, C)$  and  $(U, R)$ , which means these profiles meet the requirements for NE. Thus, we conclude this game has two pure-strategy Nash equilibria.

### 2.1.1 Diving Money

(Osborne, 38.2) Two players have \$1 to divide. Each names an integer  $0 \leq k \leq 10$ . If  $k_1 + k_2 \leq 10$ , each gets  $k_i$ . If  $k_1 + k_2 > 10$ , then (a) if  $k_1 < k_2$ , player 1 gets  $k_1$  and player 2 gets  $10 - k_1$ ; (b) if  $k_1 > k_2$ , player 1 gets  $10 - k_2$  and player 2 gets  $k_2$ ; and (c) if  $k_1 = k_2$ , each player gets \$5.

Instead of constructing  $10 \times 10$  matrix and using the procedure above, we shall employ an alternative, less cumbersome notation. We draw a coordinate system with 11 marks on each of the abscissa and the ordinate. We then identify the best responses for each player given any of the 11 possible strategies of his opponent. We mark the best responses for player 1 with a circle, and the best responses for player 2 with a smaller disc.

Looking at the plot makes clear which strategies are mutual best responses. This game has 4 Nash equilibria in pure strategies:  $(5, 5)$ ,  $(5, 6)$ ,  $(6, 5)$ , and  $(6, 6)$ . The payoffs in all of these are the same: each player gets \$5.

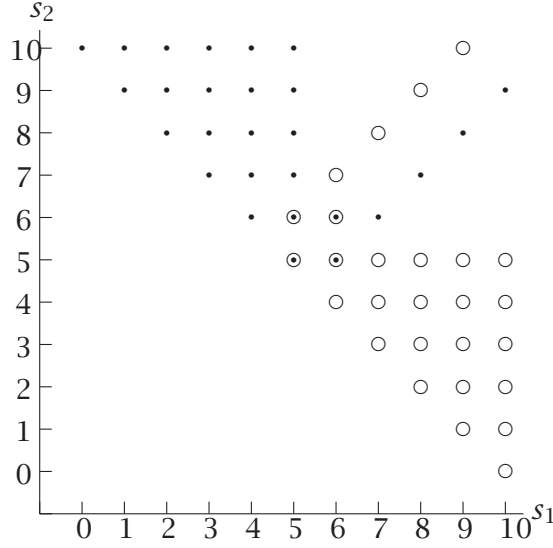


Figure 14: Best Responses in the Dividing Money Game.

### 2.1.2 The Partnership Game

There is a firm with two partners. The firm's profit depends on the effort each partner expends on the job and is given by  $p = 4(x + y + cxy)$ , where  $x$  is the amount of effort expended by partner 1 and  $y$  is the amount of effort expended by partner 2. Assume that  $x, y \in [0, 4]$ . The value  $c \in [0, 1/4]$  measures how complementary the tasks of the partners are. Partner 1 incurs a personal cost  $x^2$  of expending effort, and partner 2 incurs cost  $y^2$ . Each partner selects the level of his effort independently of the other, and both do so simultaneously. Each partner seeks to maximize their share of the firm's profit (which is split equally) net of the cost of effort. That is, the payoff function for partner 1 is  $u_1(x, y) = p/2 - x^2$ , and that for partner 2 is  $u_2(x, y) = p/2 - y^2$ .

The strategy spaces here are continuous and we cannot construct a payoff matrix. (Mathematically,  $S_1 = S_2 = [0, 4]$  and  $\Delta S = [0, 4] \times [0, 4]$ .) We can, however, analyze this game using best response functions. Let  $\hat{y}$  represent some belief partner 1 has about the other partner's effort. In this case, partner 1's payoff will be  $2(x + \hat{y} + c\hat{y}x) - x^2$ . We need to maximize this expression with respect to  $x$  (recall that we are holding partner's two strategy constant and trying to find the optimal response for partner 1 to that strategy). Taking the derivative yields  $2 + 2c\hat{y} - 2x$ . Setting the derivative to 0 and solving for  $x$  yields the best response  $BR_1(\hat{y}) = \{1 + c\hat{y}\}$ . Going through the equivalent calculations for the other partner yields his best response function  $BR_2(\hat{x}) = \{1 + c\hat{x}\}$ .

We are now looking for a strategy profile  $(x^*, y^*)$  such that  $x^* = BR_1(y^*)$  and  $y^* = BR_2(x^*)$ . (We can use equalities here because the best response functions produce single values!) To find this profile, we solve the system of equations:

$$\begin{aligned} x^* &= 1 + cy^* \\ y^* &= 1 + cx^*. \end{aligned}$$

The solution is  $x^* = y^* = 1/(1 - c)$ . Thus, this game has a unique Nash equilibrium in pure

strategies, in which both partners expend  $1/(1 - c)$  worth of effort.

### 2.1.3 Modified Partnership Game

Consider now a game similar to that in the preceding example. Let effort be restricted to the interval  $[0, 1]$ . Let  $p = 4xy$ , and let the personal costs be  $x$  and  $y$  respectively. Thus,  $u_1(x, y) = 2xy - x = x(2y - 1)$  and  $u_2(x, y) = y(2x - 1)$ . We find the best response functions for partner 1 (the other one is the same). If  $y < 1/2$ , then, since  $2y - 1 < 0$ , partner 1's best response is 0. If  $y = 1/2$ , then  $2y - 1 = 0$ , and partner 1 can choose any level of effort. If  $y > 1/2$ , then  $2y - 1 > 0$  and so partner 1's optimal response is to choose 1. This is summarized below:

$$BR_1(y) = \begin{cases} 0 & \text{if } y < 1/2 \\ [0, 1] & \text{if } y = 1/2 \\ 1 & \text{if } y > 1/2 \end{cases}$$

Since  $BR_2(x)$  is the same, we can immediately see that there are three Nash equilibria in pure strategies:  $(0, 0)$ ,  $(1, 1)$ , and  $(1/2, 1/2)$  with payoffs  $(0, 0)$ ,  $(1, 1)$ , and  $(0, 0)$  respectively. Let's plot the best response functions, just to see this result graphically in Fig. 15 (p. 12). The three discs at the points where the best response functions intersect represent the three pure-strategy Nash equilibria we found above.

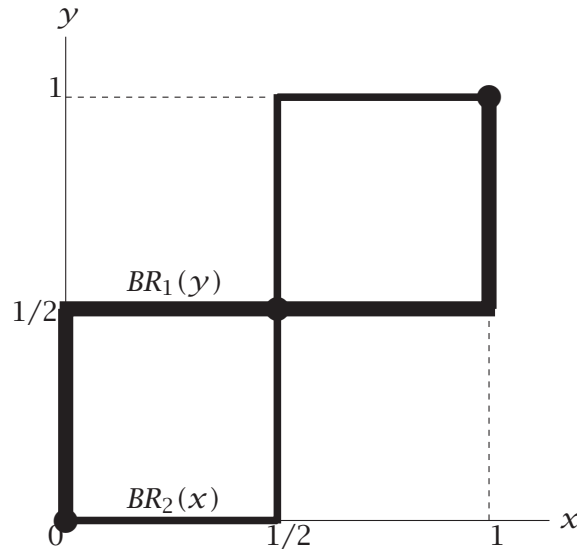


Figure 15: Best Responses in the Modified Partnership Game.

## 2.2 Strict Nash Equilibrium

Consider the game in Fig. 16 (p. 13). (Its story goes like this. The setting is the South Pacific in 1943. Admiral Kimura has to transport Japanese troops across the Bismarck Sea to New Guinea, and Admiral Kenney wants to bomb the transports. Kimura must choose between a shorter Northern route or a longer Southern route, and Kenney must decide where to send

his planes to look for the transports. If Kenney sends the plans to the wrong route, he can recall them, but the number of days of bombing is reduced.)

|        |          | Kimura   |          |
|--------|----------|----------|----------|
|        |          | <i>N</i> | <i>S</i> |
| Kenney | <i>N</i> | 2,-2     | 2,-2     |
|        | <i>S</i> | 1,-1     | 3,-3     |

Figure 16: The Battle of Bismarck Sea.

This game has a unique Nash equilibrium, in which both choose the northern route,  $(N, N)$ . Note, however, that if Kenney plays  $N$ , then Kimura is indifferent between  $N$  and  $S$  (because the advantage of the shorter route is offset by the disadvantage of longer bombing raids). Still, the strategy profile  $(N, N)$  meets the requirements of NE. This equilibrium is not strict.

More generally, an equilibrium is strict if, and only if, each player has a unique best response to the other players' strategies:

**DEFINITION 7.** A strategy profile  $(s_i^*, s_{-i}^*)$  is a **strict Nash equilibrium** if for every player  $i$ ,  $u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)$  for every strategy  $s_i \neq s_i^*$ .

The difference from the original definition of NE is only in the strict inequality sign.

### 2.3 Mixed Strategy Nash Equilibrium

The most common example of a game with no Nash equilibrium in pure strategies is MATCHING PENNIES, which is given in Fig. 17 (p. 13).

|          |          | Player 2 |          |
|----------|----------|----------|----------|
|          |          | <i>H</i> | <i>T</i> |
| Player 1 | <i>H</i> | 1, -1    | -1, 1    |
|          | <i>T</i> | -1, 1    | 1, -1    |

Figure 17: Matching Pennies.

This is a strictly competitive (zero-sum) situation, in which the gain for one player is the loss of the other.<sup>2</sup> This game has no Nash equilibrium in pure strategies. Let's consider mixed strategies.

We first extend the idea of best responses to mixed strategies: Let  $BR_i(\sigma_{-i})$  denote player  $i$ 's best response correspondence when the others play  $\sigma_{-i}$ . The definition of Nash equilibrium is analogous to the pure-strategy case:

**DEFINITION 8.** A mixed strategy profile  $\sigma^*$  is a **mixed-strategy Nash equilibrium** if, and only if,  $\sigma_i^* \in BR_i(\sigma_{-i}^*)$ .

As before, a strategy profile is a Nash equilibrium whenever all players' strategies are best responses to each other. For a mixed strategy to be a best response, it must put positive probabilities only on pure strategies that are best responses. Mixed strategy equilibria, like pure strategy equilibria, never use dominated strategies.

<sup>2</sup>It is these zero-sum games that von Neumann and Morgenstern studied and found solutions for. However, Nash's solution can be used in non-zero-sum games, and is thus far more general and useful.

Turning now to Matching Pennies, let  $\sigma_1 = (p, 1 - p)$  denote a mixed strategy for player 1 where he chooses  $H$  with probability  $p$ , and  $T$  with probability  $1 - p$ . Similarly, let  $\sigma_2 = (q, 1 - q)$  denote a mixed strategy for player 2 where she chooses  $H$  with probability  $q$ , and  $T$  with probability  $1 - q$ . We now derive the best response correspondence for player 1 as a function of player 2's mixed strategy.

Player 1's expected payoffs from his pure strategies given player 2's mixed strategy are:

$$\begin{aligned} u_1(H, \sigma_2) &= (1)q + (-1)(1 - q) = 2q - 1 \\ u_1(T, \sigma_2) &= (-1)q + (1)(1 - q) = 1 - 2q. \end{aligned}$$

Playing  $H$  is a best response if, and only if:

$$\begin{aligned} u_1(H, \sigma_2) &\geq u_1(T, \sigma_2) \\ 2q - 1 &\geq 1 - 2q \\ q &\geq 1/2. \end{aligned}$$

Analogously,  $T$  is a best response if, and only if,  $q \leq 1/2$ . Thus, player 1 should choose  $p = 1$  if  $q \geq 0.5$  and  $p = 0$  if  $q \leq 0.5$ . Note now that whenever  $q = 0.5$ , player 1 is indifferent between his two pure strategies: choosing either one yields the same expected payoff of 0. Thus, both strategies are best responses, which implies that any mixed strategy that includes both of them in its support is a best response as well. Again, the reason is that if the player is getting the same expected payoff from his two pure strategies, he will get the same expected payoff from any mixed strategy whose support they are.

Analogous calculations yield the best response correspondence for player 2 as a function of  $\sigma_1$ . Putting these together yields:

$$BR_1(q) = \begin{cases} 0 & \text{if } q < 1/2 \\ [0, 1] & \text{if } q = 1/2 \\ 1 & \text{if } q > 1/2 \end{cases} \quad BR_2(p) = \begin{cases} 0 & \text{if } p > 1/2 \\ [0, 1] & \text{if } p = 1/2 \\ 1 & \text{if } p < 1/2 \end{cases}$$

The graphical representation of the best response correspondences is in Fig. 18 (p. 15). The only place where the randomizing strategies are best responses to each other is at the intersection point, where each player randomizes between the two strategies with probability  $1/2$ . Thus, the Matching Pennies game has a unique Nash equilibrium in mixed strategies  $\langle \sigma_1^*, \sigma_2^* \rangle$ , where  $\sigma_1^* = (1/2, 1/2)$ , and  $\sigma_2^* = (1/2, 1/2)$ . That is, where  $p = q = 0.5$ .

As before, the alternative definition of Nash equilibrium is in terms of the payoff functions. We require that no player can do better by using any other strategy than the one he uses in the equilibrium mixed strategy profile given that all other players stick to their mixed strategies. In other words, the player's expected payoff of the MSNE profile is at least as good as the expected payoff of using any other strategy.

**DEFINITION 9.** A mixed strategy profile  $\sigma^*$  is a **mixed-strategy Nash equilibrium** if, for all players  $i$ ,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \text{ for all } s_i \in S_i.$$

Since expected utilities are linear in the probabilities, if a player uses a non-degenerate mixed strategy in a Nash equilibrium, then he must be indifferent between all pure strategies to which he assigns positive probability. This is why we only need to check for a profitable pure strategy deviation. (Note that this differs from Osborne's definition, which involves checking against profitable mixed strategy deviations.)

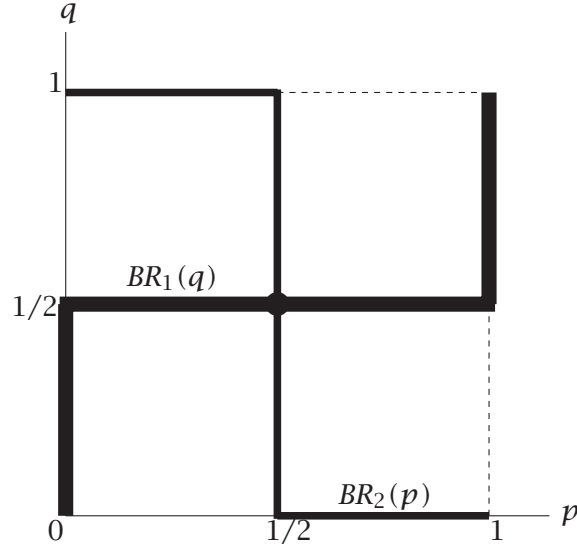


Figure 18: Best Responses in Matching Pennies.

### 2.3.1 Battle of the Sexes

We now analyze the Battle of the Sexes game, reproduced in Fig. 19 (p. 15).

|          |          | Player 2 |          |
|----------|----------|----------|----------|
|          |          | <i>F</i> | <i>B</i> |
| Player 1 | <i>F</i> | 2,1      | 0,0      |
|          | <i>B</i> | 0,0      | 1,2      |

Figure 19: Battle of the Sexes.

As a first step, we plot each player's expected payoff from each of the pure strategies as a function of the other player's mixed strategy. Let  $p$  denote the probability that player 1 chooses  $F$ , and let  $q$  denote the probability that player 2 chooses  $F$ . Player 1's expected payoff from  $F$  is then  $2q + 0(1 - q) = 2q$ , and his payoff from  $B$  is  $0q + 1(1 - q) = 1 - q$ . Since  $2q = 1 - q$  whenever  $q = 1/3$ , the two lines intersect there.

Looking at the plot in Fig. 20 (p. 16) makes it obvious that for any  $q < 1/3$ , player 1 has a unique best response in playing the pure strategy  $B$ , for  $q > 1/3$ , his best response is again unique and it is the pure strategy  $F$ , while at  $q = 1/3$ , he is indifferent between his two pure strategies, which also implies he will be indifferent between any mixing of them. Thus, we can specify player 1's best response (in terms of  $p$ ):

$$BR_1(q) = \begin{cases} 0 & \text{if } q < 1/3 \\ [0, 1] & \text{if } q = 1/3 \\ 1 & \text{if } q > 1/3 \end{cases}$$

We now do the same for the expected payoffs of player 2's pure strategies as a function of player 1's mixed strategy. Her expected payoff from  $F$  is  $1p + 0(1 - p) = p$  and her expected

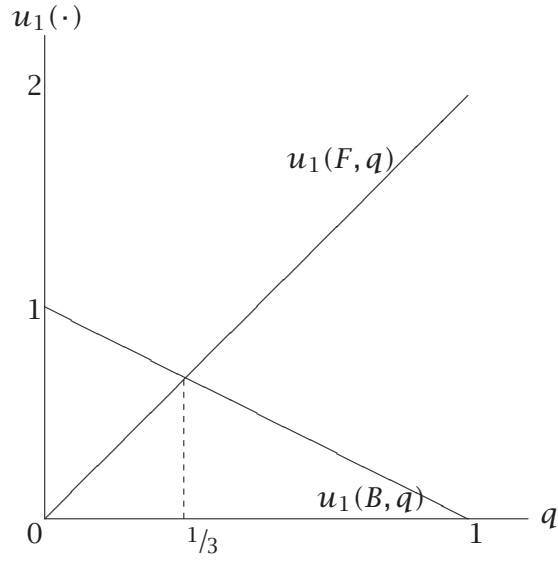


Figure 20: Player 1 Expected Payoffs as a Function of the Player 2s Mixed Strategy.

payoff from  $B$  is  $0p + 2(1 - p) = 2(1 - p)$ . Noting that  $p = 2(1 - p)$  whenever  $p = 2/3$ , we should expect that the plots of her expected payoffs from the pure strategies will intersect at  $p = 2/3$ . Indeed, Fig. 21 (p. 16) shows that this is the case.

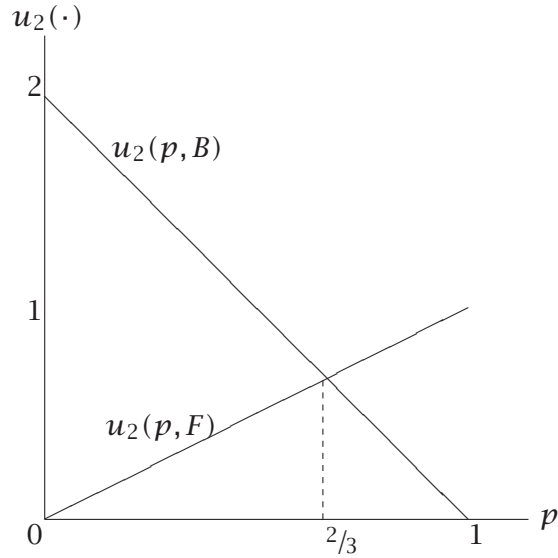


Figure 21: Player 2's Expected Payoffs as a Function of Player 1's Mixed Strategy.

Looking at the plot reveals that player 2 strictly prefers playing  $B$  whenever  $p < 2/3$ , strictly prefers playing  $F$  whenever  $p > 2/3$ , and is indifferent between the two (and any mixture of

them) whenever  $p = 2/3$ . This allows us to specify her best response (in terms of  $q$ ):

$$BR_2(p) = \begin{cases} 0 & \text{if } p < 2/3 \\ [0, 1] & \text{if } p = 2/3 \\ 1 & \text{if } p > 2/3 \end{cases}$$

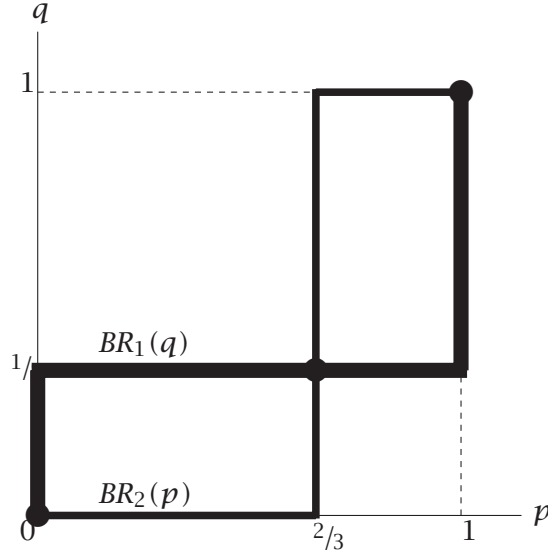


Figure 22: Best Responses in Battle of the Sexes.

Having derived the best response correspondences, we can plot them in the  $p \times q$  space, which is done in Fig. 22 (p. 17). The best response correspondences intersect in three places, which means there are three mixed strategy profiles in which the two strategies are best responses of each other. Two of them are in pure-strategies: the degenerate mixed strategy profiles  $\langle 1, 1 \rangle$  and  $\langle 0, 0 \rangle$ . In addition, there is one mixed-strategy equilibrium,

$$\langle (2/3[F], 1/3[B]), (1/3[F], 2/3[B]) \rangle.$$

In the mixed strategy equilibrium, each outcome occurs with positive probability. To calculate the corresponding probability, multiply the equilibrium probabilities of each player choosing the relevant action. This yields  $\Pr(F, F) = 2/3 \times 1/3 = 2/9$ ,  $\Pr(B, B) = 1/3 \times 2/3 = 2/9$ ,  $\Pr(F, B) = 2/3 \times 2/3 = 4/9$ , and  $\Pr(B, F) = 1/3 \times 1/3 = 1/9$ . Thus, player 1 and player 2 will meet with probability  $4/9$  and fail to coordinate with probability  $5/9$ . Obviously, these probabilities have to sum up to 1. Both players' expected payoff from this equilibrium is  $(2)2/9 + (1)2/9 = 2/3$ .

## 2.4 Computing Nash Equilibria

Remember that a mixed strategy  $\sigma_i$  is a best response to  $\sigma_{-i}$  if, and only if, every pure strategy in the support of  $\sigma_i$  is itself a best response to  $\sigma_{-i}$ . Otherwise player  $i$  would be able to improve his payoff by shifting probability away from any pure strategy that is not a best response to any that is.

This further implies that in a mixed strategy Nash equilibrium, where  $\sigma_i^*$  is a best response to  $\sigma_{-i}^*$  for all players  $i$ , all pure strategies in the support of  $\sigma_i^*$  yield the same payoff when played against  $\sigma_{-i}^*$ , and no other strategy yields a strictly higher payoff. We now use these remarks to characterize mixed strategy equilibria.

REMARK 4. In any finite game, for every player  $i$  and a mixed strategy profile  $\sigma$ ,

$$u_i(\sigma) = \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i}).$$

That is, the player's payoff to the mixed strategy profile is the weighted average of his expected payoffs to all mixed strategy profiles where he plays every one of his pure strategies with a probability specified by his mixed strategy  $\sigma_i$ .

For example, returning to the BoS game, consider the strategy profile  $(1/4, 1/3)$ . Player 1's expected payoff from this strategy profile is:

$$\begin{aligned} u_1(1/4, 1/3) &= (1/4) u_1(F, 1/3) + (3/4) u_1(B, 1/3) \\ &= (1/4) [(2)^{1/3} + (0)^{2/3}] + (3/4) [(0)^{1/3} + (1)^{2/3}] \\ &= 2/3 \end{aligned}$$

The property in Remark 4 allows us to check whether a mixed strategy profile is an equilibrium by examining each player's expected payoffs to his pure strategies only. (Recall that the definition of MSNE I gave you is actually stated in precisely these terms.)

PROPOSITION 1. *For any finite game, a mixed strategy profile  $\sigma^*$  is a mixed strategy Nash equilibrium if, and only if, for each player  $i$*

1.  $u_i(s_i, \sigma_{-i}^*) = u_i(s_j, \sigma_{-i}^*)$  for all  $s_i, s_j \in \text{supp}(\sigma_i^*)$
2.  $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_k, \sigma_{-i}^*)$  for all  $s_i \in \text{supp}(\sigma_i^*)$  and all  $s_k \notin \text{supp}(\sigma_i^*)$ .

That is, the strategy profile  $\sigma^*$  is a MSNE if for every player, the payoff from any pure strategy in the support of his mixed strategy is the same, and at least as good as the payoff from any pure strategy not in the support of his mixed strategy when all other players play their MSNE mixed strategies. In other words, **if a player is randomizing in equilibrium, he must be indifferent among all pure strategies in the support of his mixed strategy.** It is easy to see why this must be the case by supposing that it must not. If he player is not indifferent, then there is at least one pure strategy in the support of his mixed strategy that yields a payoff strictly higher than some other pure strategy that is also in the support. If the player deviates to a mixed strategy that puts a higher probability on the pure strategy that yields a higher payoff, he will strictly increase his expected payoff, and thus the original mixed strategy cannot be optimal; i.e. it cannot be a strategy he uses in equilibrium.

Clearly, a Nash equilibrium that involves mixed strategies cannot be strict because if a player is willing to randomize in equilibrium, then he must have more than one best response. In other words, strict Nash equilibria are always in pure strategies.

We also have a very useful result analogous to the one that states that no player uses a strictly dominated strategy in equilibrium. That is, a dominated strategy is never a best response to any combination of mixed strategies of the other players.

PROPOSITION 2. *A strictly dominated strategy is not used with positive probability in any mixed strategy equilibrium.*

*Proof.* Let  $s_i$  be a pure strategy that is strictly dominated by the mixed strategy  $\sigma_i$ , and let  $\sigma_{-i}$  be the other players' mixed strategies. Player  $i$ 's expected payoff from his mixed strategy is  $u_i(\sigma_i, \sigma_{-i})$  and his expected payoff from his pure strategy is  $u_i(s_i, \sigma_{-i})$ . The expected payoffs are weighted averages of the payoffs from  $u_i(\cdot, s_{-i})$ , for all  $s_{-i} \in S_{-i}$  with the weights assigned by  $\sigma_{-i}$  (that is, the weights are the probabilities with which different profiles occur). But since  $s_i$  is strictly dominated by  $\sigma_i$ , it follows that  $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$  for all  $s_{-i} \in S_{-i}$ .  $\square$

This means that when we are looking for mixed strategy equilibria, we can eliminate from consideration all strictly dominated strategies. It is important to note that, as in the case of pure strategies, *we cannot eliminate weakly dominated strategies from consideration* when finding mixed strategy equilibria (because a weakly dominated strategy *can* be used with positive probability in a MSNE).

#### 2.4.1 Myerson's Card Game

The strategic form of the game is given in Fig. 7 (p. 6). It is easy to verify that there are no equilibria in pure strategies. Further, as we have shown, the strategy  $Ff$  is strictly dominated, so we can eliminate it from the analysis. The resulting game is shown in Fig. 23 (p. 19).

|          |      | Player 2  |       |
|----------|------|-----------|-------|
|          |      | $m$       | $p$   |
| Player 1 | $Rr$ | 0, 0      | 1, -1 |
|          | $Rf$ | -0.5, 0.5 | 1, -1 |
|          | $Fr$ | 0.5, -0.5 | 0, 0  |

Figure 23: The Reduced Strategic Form of the Myerson Card Game.

Let  $q$  denote the probability with which player 2 chooses  $m$ , and  $1 - q$  be the probability with which she chooses  $p$ . We now show that in equilibrium player 1 would not play  $Rf$  with positive probability. The expected payoff from  $Rf$  is  $u_1(Rf, q) = (-0.5)q + (1)(1 - q) = 1 - 1.5q$ , while the expected payoff from  $Rr$  is  $u_1(Rr, q) = (0)q + (1)(1 - q) = 1 - q$ . Suppose that in equilibrium player 1 puts positive probability on the pure strategy  $Rf$ . This implies  $1 - 1.5q \geq 1 - q$ , which implies that  $q = 0$ , that is, player 2 must be choosing  $p$  with certainty. If this is the case, then player 1 cannot be putting positive probability on  $Fr$  in equilibrium because it yields a payoff of 0 while the other two strategies yield 1. Thus, if player 1 puts positive probability on  $Rf$  in equilibrium, he must be mixing only between  $Rr$  and  $Rf$ . Player 2's expected utility from  $m$  in this case is  $u_2(\sigma_1, m) = \sigma_1(Rr)(0) + \sigma_1(Rf)(0.5) = (0.5)\sigma_1(Rf)$ , and her expected utility from  $p$  is  $u_2(\sigma_1, p) = \sigma_1(Rr)(-1) + \sigma_1(Rf)(-1) = -\sigma_1(Rr) - \sigma_1(Rf)$ . Because  $q = 0$ , it follows that her payoff from  $p$  must be at least as good as the payoff from  $m$ , or:

$$\begin{aligned} (0.5)\sigma_1(Rf) &\leq -\sigma_1(Rr) - \sigma_1(Rf) \\ (1.5)\sigma_1(Rf) &\leq -\sigma_1(Rr), \end{aligned}$$

which is a contradiction because  $\sigma_1(Rf) > 0$ . Therefore, it cannot be the case that player 1 puts positive probability on  $Rf$  in equilibrium.

We conclude that any Nash equilibrium must involve player 1 mixing between  $Rr$  and  $Fr$ . Let  $s$  be the probability of choosing  $Rr$ , and  $1 - s$  be the probability of choosing  $Fr$ . Because

player 1 is willing to mix, the expected payoffs from the two pure strategies must be equal. Thus,  $(0)q + (1)(1 - q) = (0.5)q + (0)(1 - q)$ , which implies that  $q = 2/3$ . Since player 2 must be willing to randomize as well, her expected payoffs from the pure strategies must also be equal. Thus,  $(0)s + (-0.5)(1 - s) = (-1)s + (0)(1 - s)$ , which implies that  $s = 1/3$ . We conclude that the unique mixed strategy Nash equilibrium of the card game is

$$\langle (1/3[Rr], 2/3[Fr]), (2/3[m], 1/3[p]) \rangle.$$

That is, player 1 raises for sure if he has a red (winning) card, and raises with probability  $1/3$  if he has a black (losing) card. Player 2 meets with probability  $2/3$  when she sees player 1 raise in equilibrium. The expected utility payoff in this unique equilibrium for player 1 is:

$$(0.5) [2/3(2) + 1/3(1)] + (0.5) [1/3 (2/3(-2) + 1/3(1)) + 2/3(-1)] = 1/3,$$

and the expected payoff for player 2, computed analogously, is  $-1/3$ . If you are risk-neutral, you should only agree to take player 2's role if offered a pre-play bribe of at least \$0.34 because you expect to lose \$0.33.

#### 2.4.2 Another Simple Game

To illustrate the algorithm for solving strategic form games, we now go through a detailed example using the game from Myerson, p. 101, reproduced in Fig. 24 (p. 20). The algorithm for finding all Nash equilibria involves (a) checking for solutions in pure strategies, and (b) checking for solutions in mixed strategies. Step (b) is usually the more complicated one, especially when there are many pure strategies to consider. You will need to make various guesses, use insights from dominance arguments, and utilize the remarks about optimal mixed strategies here.

|          |          | Player 2 |          |          |
|----------|----------|----------|----------|----------|
|          |          | <i>L</i> | <i>M</i> | <i>R</i> |
| Player 1 | <i>U</i> | 7, 2     | 2, 7     | 3, 6     |
|          | <i>D</i> | 2, 7     | 7, 2     | 4, 5     |

Figure 24: A Strategic Form Game.

We begin by looking for pure-strategy equilibria. *U* is only a best response to *L*, but the best response to *U* is *M*. There is no pure-strategy equilibrium involving player 1 choosing *U*. On the other hand, *D* is a best response to both *M* and *R*. However, only *L* is a best response to *D*. Therefore, there is no pure-strategy equilibrium with player 1 choosing *D* for sure. This means that any equilibrium must involve a mixed strategy for player 1 with  $\text{supp}(\sigma_1) = \{U, D\}$ .

Turning now to player 2's strategy, we note that there can be no equilibrium with player 2 choosing a pure strategy either. This is because player 1 has a unique best response to each of her three strategies, but we have just seen that player 1 must be randomizing in equilibrium.

We now have to make various guesses about the support of player 2's strategy. We know that it must include at least two of her pure strategies, and perhaps all three. There are four possibilities to try.

- $\text{supp}(\sigma_2) = \{L, M, R\}$ . Since player 2 is willing to mix, she must be indifferent between her pure strategies, and therefore:

$$2\sigma_1(U) + 7\sigma_1(D) = 7\sigma_1(U) + 2\sigma_1(D) = 6\sigma_1(U) + 5\sigma_1(D).$$

We require that the mixture is a valid probability distribution, or  $\sigma_1(U) + \sigma_1(D) = 1$ .

Note now that  $2\sigma_1(U) + 7\sigma_1(D) = 7\sigma_1(U) + 2\sigma_1(D) \Rightarrow \sigma_1(U) = \sigma_1(D) = 0.5$ . However,  $7\sigma_1(U) + 2\sigma_1(D) = 6\sigma_1(U) + 5\sigma_1(D) \Rightarrow \sigma_1(U) = 3\sigma_1(D)$ , a contradiction. Therefore, there can be no equilibrium that includes all three of player 2's strategies in the support of her mixed strategy.

- $\text{supp}(\sigma_2) = \{M, R\}$ . Since player 1 is willing to mix, it must be the case that  $2\sigma_2(M) + 3\sigma_2(R) = 7\sigma_2(M) + 4\sigma_2(R) \Rightarrow 0 = 5\sigma_2(M) + \sigma_2(R)$ , which is clearly impossible because both  $\sigma_2(M) > 0$  and  $\sigma_2(R) > 0$ . Hence, there can be no equilibrium where player 2's support consists of  $M$  and  $R$ .
- $\text{supp}(\sigma_2) = \{L, M\}$ . Because player 1 is willing to mix, it follows that  $7\sigma_2(L) + 2\sigma_2(M) = 2\sigma_2(L) + 7\sigma_2(M) \Rightarrow \sigma_2(L) = \sigma_2(M) = 0.5$ . Further, because player 2 is willing to mix, it follows that  $2\sigma_1(U) + 7\sigma_1(D) = 7\sigma_1(U) + 2\sigma_1(D) \Rightarrow \sigma_1(U) = \sigma_1(D) = 0.5$ .

So far so good. We now check for profitable deviations. If player 1 is choosing each strategy with positive probability, then choosing  $R$  would yield player 2 an expected payoff of  $(0.5)(6) + (0.5)(5) = 5.5$ . This must be worse than any of the strategies in the support of her mixed strategy, so let's check  $M$ . Her expected payoff from  $M$  is  $(0.5)(7) + (0.5)(2) = 5$ . That is, the strategy to which she assigns positive probability yields an expected payoff strictly higher than any of the strategies in the support of her mixed strategy. Therefore, this cannot be an equilibrium either.

- $\text{supp}(\sigma_2) = \{L, R\}$ . Since player 1 is willing to mix, it follows that  $7\sigma_2(L) + 3\sigma_2(R) = 2\sigma_2(L) + 4\sigma_2(R) \Rightarrow 5\sigma_2(L) = \sigma_2(R)$ , which in turn implies  $\sigma_2(L) = 1/6$ , and  $\sigma_2(R) = 5/6$ . Further, since player 2 is willing to mix, it follows that  $2\sigma_1(U) + 7\sigma_1(D) = 6\sigma_1(U) + 5\sigma_1(D) \Rightarrow \sigma_1(D) = 2\sigma_1(U)$ , which in turn implies  $\sigma_1(U) = 1/3$ , and  $\sigma_1(D) = 2/3$ .

Can player 2 do better by choosing  $M$ ? Her expected payoff would be  $(1/3)(7) + (2/3)(2) = 11/3$ . Any of the pure strategies in the support of her mixed strategy yields an expected payoff of  $(1/3)(2) + (2/3)(7) = (1/3)(6) + (2/3)(5) = 16/3$ , which is strictly better. Therefore, the mixed strategy profile:

$$\langle (\sigma_1(U) = 1/3, \sigma_1(D) = 2/3), (\sigma_2(L) = 1/6, \sigma_2(R) = 5/6) \rangle$$

is the unique Nash equilibrium of this game. The expected equilibrium payoffs are  $11/3$  for player 1 and  $16/3$  for player 2.

This exhaustive search for equilibria may become impractical when the games become larger (either more players or more strategies per player). There are programs, like the late Richard McKelvey's *Gambit*, that can search for solutions to many games.

### 2.4.3 Choosing Numbers

Players 1 and 2 each choose a positive integer up to  $K$ . Thus, the strategy spaces are both  $\{1, 2, \dots, K\}$ . If the players choose the same number then player 2 pays \$1 to player 1,

otherwise no payment is made. Each player's preferences are represented by his expected monetary payoff. The claim<sup>3</sup> is that the game has a mixed strategy Nash equilibrium in which each player chooses each positive integer with equal probability.

It is easy to see that this game has no equilibrium in pure strategies: If the strategy profile specifies the same numbers, then player 2 can profitably deviate to any other number; if the strategy profile specifies different numbers, then player 1 can profitably deviate to the number that player 2 is naming. However, this is a finite game, and so Nash's Theorem tells us there must be an equilibrium. Thus, we know we should be looking for one in mixed strategies.

The problem here is that there is an infinite number of potential mixtures we have to consider. We attack this problem methodically by looking at types of mixtures instead of individual ones. One focal mixture is where each number is chosen with the same probability. Since there are  $K$  numbers, this probability is  $1/K$ .

We now apply Proposition 1. Since all strategies are in the support of this mixed strategy, it is sufficient to show that each strategy of each player results in the same expected payoff. (That is, we only use the first part of the proposition.) Player 1's expected payoff from each pure strategy is  $1/K(1) + (1 - 1/K)(0) = 1/K$  because player 2 chooses the same number with probability  $1/K$  and a different number with the complementary probability. Similarly, player 2's expected payoff is  $1/K(-1) + (1 - 1/K)(0) = -1/K$ . Thus, this strategy profile is a mixed strategy Nash equilibrium.

Is this the only MSNE? Let  $(\sigma_1^*, \sigma_2^*)$  be a MSNE where  $\sigma_i^*(k)$  is the probability that player  $i$ 's mixed strategy assigns to the integer  $k$ . Given that player 2 uses  $\sigma_2^*$ , player 1's expected payoff to choosing the number  $k$  is  $\sigma_2^*(k)$ . From Proposition 1, if  $\sigma_1^*(k) > 0$ , then  $\sigma_2^*(k) \geq \sigma_2^*(j)$  for all numbers  $j$ . That is, if player 1 assigns positive probability to choosing some number  $k$ , then the equilibrium probability with which player 2 is choosing this number must be at least as great as the probability of any other number. If this were not the case, it would mean that there exists some number  $m \neq k$  which player 2 chooses with a higher probability in equilibrium. But in that case, player 1's equilibrium strategy would be strictly dominated by the strategy that chooses  $m$  with a higher probability (because it would yield player 1 a higher expected payoff). Therefore, the mixed strategy  $\sigma_1^*$  could not be optimal, a contradiction.

Since  $\sigma_2^*(j) > 0$  for at least some  $j$ , it follows that  $\sigma_2^*(k) > 0$ . That is, because player 2 must choose some number, she must assign a strictly positive probability to at least one number from the set. But because the equilibrium probability of choosing  $k$  must be at least as high as the probability of any other number, the probability of  $k$  must be strictly positive.

We conclude that in equilibrium, if player 1 assigns positive probability to some arbitrary number  $k$ , then player 2 must do so as well.

Now, player 2's expected payoff if she chooses  $k$  is  $-\sigma_1^*(k)$ , and since  $\sigma_2^*(k) > 0$ , it must be the case that  $\sigma_1^*(k) \leq \sigma_1^*(j)$  for all  $j$ . This follows from Proposition 1. To see this, note that if this did not hold, it would mean that player 1 is choosing some number  $m$  with a strictly lower probability in equilibrium. However, in this case player 2 could do strictly better by switching to a strategy that picks  $m$  because the expected payoff would improve (the numbers are less likely to match). But this contradicts the optimality of player 2's equilibrium strategy  $\sigma_2^*$ .

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<sup>3</sup>It is not clear how you get to this claim. This is the part of game theory that often requires some inspired guesswork and is usually the hardest part. Once you have an idea about an equilibrium, you can check whether the profile is one. There is usually no mechanical way of finding an equilibrium.

What is the largest equilibrium probability with which  $k$  is chosen by player 1? We know that it cannot exceed the probability assigned to any other number. Because there are  $K$  numbers, this means that it cannot exceed  $1/K$ . To see this, note that if there was some number to which the assigned probability was strictly greater than  $1/K$ , then there must be some other number with a probability strictly smaller than  $1/K$ , and then  $\sigma_1^*(k)$  would have to be no greater than that smaller probability. We conclude that  $\sigma_1^*(k) \leq 1/K$ .

We have now shown that if in equilibrium player 1 assigns positive probability to some arbitrary number  $k$ , it follows that this probability cannot exceed  $1/K$ . Hence, the equilibrium probability of choosing any number to which player 1 assigns positive probability cannot exceed  $1/K$ .

But this now implies that player 1 must assign  $1/K$  to each number and mix over all available numbers. Suppose not, which would mean that player 1 is mixing over  $n < K$  numbers. From the proof above, we know that he cannot assign more than  $1/K$  probability to each of these  $n$  numbers. But because his strategy must be a valid probability distribution, the individual probabilities must sum up to 1. In this case, the sum up to  $n/K < 1$  because  $n < K$ . The only way to meet the requirement would be to assign at least one of the numbers a strictly larger probability, a contradiction. Therefore,  $\sigma_1^*(k) = 1/K$  for all  $k$ .

A symmetric argument establishes the result for player 2. We conclude that there are no other mixed strategy Nash equilibria in this game.

#### 2.4.4 Defending Territory

General A is defending territory accessible by 2 mountain passes against General B. General A has 3 divisions at his disposal and B has 2. Each must allocate divisions between the two passes. A wins the pass if he allocates at least as many divisions to it as B does. A successfully defends his territory if he wins at both passes.

General A has four strategies at his disposal, depending on the number of divisions he allocates to each pass:  $S_A = \{(3, 0), (2, 1), (1, 2), (0, 3)\}$ . General B has three strategies he can use:  $S_B = \{(2, 0), (1, 1), (0, 2)\}$ . We construct the payoff matrix as shown in Fig. 25 (p. 23).

|           |        | General B |        |        |
|-----------|--------|-----------|--------|--------|
|           |        | (2, 0)    | (1, 1) | (0, 2) |
| General A | (3, 0) | 1, -1     | -1, 1  | -1, 1  |
|           | (2, 1) | 1, -1     | 1, -1  | -1, 1  |
|           | (1, 2) | -1, 1     | 1, -1  | 1, -1  |
|           | (0, 3) | -1, 1     | -1, 1  | 1, -1  |

Figure 25: Defending Territory.

This is a strictly competitive game, which (not surprisingly) has no pure strategy Nash equilibrium. Thus, we shall be looking for MSNE. Denote a mixed strategy of General A by  $(p_1, p_2, p_3, p_4)$ , and a mixed strategy of General B by  $(q_1, q_2, q_3)$ .

First, suppose that in equilibrium  $q_2 > 0$ . Since General A's expected payoff from his strategies (3, 0) and (0, 3) are both less than any of the other two strategies, it follows that in such an equilibrium  $p_1 = p_4 = 0$ . In this case, General B's expected payoff to his strategy (1, 1) is then  $-1$ . However, either one of the other two available strategies would yield a higher expected payoff. Therefore,  $q_2 > 0$  cannot occur in equilibrium.

Given that in any equilibrium  $q_2 = 0$ , what probabilities would B assign to the other two strategies in equilibrium? Since  $q_2 = 0$ , it follows that  $q_3 = 1 - q_1$ . General A's expected payoff to (3, 0) and (2, 1) is  $2q_1 - 1$ , and the payoff to (1, 2) and (0, 3) is  $1 - 2q_1$ . If  $q_1 < 1/2$ , then in any equilibrium  $p_1 = p_2 = 0$ . In this case, B has a unique best response, which is (2, 0), which implies that in equilibrium  $q_1 = 1$ . But if this is the case, then either of A's strategies (3, 0) or (2, 1) yields a higher payoff than any of the other two, contradicting  $p_1 = p_2 = 0$ . Thus,  $q_1 < 1/2$  cannot occur in equilibrium. Similarly,  $q_1 > 1/2$  cannot occur in equilibrium. This leaves  $q_1 = q_3 = 1/2$  to consider.

If  $q_1 = q_3 = 1/2$ , then General A's expected payoffs to all his strategies are equal. We now have to check whether General B's payoffs from this profile meet the requirements of Proposition 1. That is, we have to check whether the payoffs from (2, 0) and (0, 2) are the same, and whether this payoff is at least as good as the one to (1, 1). The first condition is:

$$\begin{aligned} -p_1 - p_2 + p_3 + p_4 &= p_1 + p_2 - p_3 - p_4 \\ p_1 + p_2 &= p_3 + p_4 = 1/2 \end{aligned}$$

General B's expected payoff to (2, 0) and (0, 2) is then 0, so the first condition is met. Note now that since  $p_1 + p_2 + p_3 + p_4 = 1$ , we have  $1 - (p_1 + p_4) = p_2 + p_3$ . The second condition is:

$$\begin{aligned} p_1 - p_2 - p_3 + p_4 &\leq 0 \\ p_1 + p_4 &\leq p_2 + p_3 \\ p_1 + p_4 &\leq 1 - (p_1 + p_4) \\ p_1 + p_4 &\leq 1/2 \end{aligned}$$

Thus, we conclude that the set of mixed strategy Nash equilibria in this game is the set of strategy profiles:

$$((p_1, 1/2 - p_1, 1/2 - p_4, p_4), (1/2, 0, 1/2)) \text{ where } p_1 + p_4 \leq 1/2.$$

#### 2.4.5 Choosing Two-Thirds of the Average

(Osborne, 34.1) Each of 3 players announces an integer from 1 to  $K$ . If the three integers are different, the one whose integer is closest to  $2/3$  of the average of the three wins \$1. If two or more integers are the same, \$1 is split equally between the people whose integers are closest to  $2/3$  of the average.

Formally,  $N = \{1, 2, 3\}$ ,  $S_i = \{1, 2, \dots, K\}$ , and  $\Delta S = S_1 \times S_2 \times S_3$ . There are  $K^3$  different strategy profiles to examine, so instead we analyze *types* of profiles.

Suppose all three players announce the same number  $k \geq 2$ . Then  $2/3$  of the average is  $2/3k$ , and each gets  $\$1/3$ . Suppose now one of the players deviates to  $k - 1$ . Now  $2/3$  of the average is  $2/3k - 2/9$ . We now wish to show that the player with  $k - 1$  is closer to the new  $2/3$  of the average than the two whose integers were  $k$ :

$$\begin{aligned} 2/3k - 2/9 - (k - 1) &< k - (2/3k - 2/9) \\ k &> 5/6 \end{aligned}$$

Since  $k \geq 2$ , the inequality is always true. Therefore, the player with  $k - 1$  is closer, and thus he can get the entire \$1. We conclude that for any  $k \geq 2$ , the profile  $(k, k, k)$  cannot be a Nash equilibrium.

The strategy profile  $(1, 1, 1)$ , on the other hand, is NE. (Note that the above inequality works just fine for  $k = 1$ . However, since we cannot choose 0 as the integer, it is not possible to undercut the other two players with a smaller number.)

We now consider an strategy profile where not all three integers are the same. First consider a profile, in which one player names a highest integer. Denote an arbitrary such profile by  $(k^*, k_1, k_2)$ , where  $k^*$  is the highest integer and  $k_1 \geq k_2$ . Two thirds of the average for this profile is  $a = \frac{2}{9}(k^* + k_1 + k_2)$ . If  $k_1 > a$ , then  $k^*$  is further from  $a$  than  $k_1$ , and therefore  $k^*$  does not win anything. If  $k_1 < a$ , then the difference between  $k^*$  and  $a$  is  $k^* - a = \frac{7}{9}k^* - \frac{2}{9}k_1 - \frac{2}{9}k_2$ . The difference between  $k_1$  and  $a$  is  $a - k_1 = \frac{2}{9}k^* - \frac{7}{9}k_1 + \frac{2}{9}k_2$ . The difference between the two is then  $\frac{5}{9}k^* + \frac{5}{9}k_1 - \frac{4}{9}k_2 > 0$ , and so  $k_1$  is closer to  $a$ . Thus  $k^*$  does not win and the player who offers it is better off by deviating to  $k_1$  and sharing the prize. Thus, no profile in which one player names a highest integer can be Nash equilibrium.

Consider now a profile in which two players name highest integers. Denote this profile by  $(k^*, k^*, k)$  with  $k^* > k$ . Then  $a = \frac{4}{9}k^* + \frac{2}{9}k$ . The midpoint of the difference between  $k^*$  and  $k$  is  $\frac{1}{2}(k^* + k) > a$ . Therefore,  $k$  is closer to  $a$  and wins the entire \$1. Either of the two other players can deviate by switching to  $k$  and thus share the prize. Thus, no such profile can be Nash equilibrium.

This exhausts all possible strategy profiles. We conclude that this game has a unique Nash equilibrium, in which all three players announce the integer 1.

#### 2.4.6 Voting for Candidates

(Osborne, 34.2) There are  $n$  voters, of which  $k$  support candidate A and  $m = n - k$  support candidate B. Each voter can either vote for his preferred candidate or abstain. Each voter gets a payoff of 2 if his preferred candidate wins, 1 if the candidates tie, and 0 if his candidate loses. If the citizen votes, he pays a cost  $c \in (0, 1)$ .

- What is the game with  $m = k = 1$ ?
- Find the pure-strategy Nash equilibria for  $k = m$ .
- Find the pure-strategy Nash equilibria for  $k < m$ .

We tackle each part in turn:

- Let's draw the bi-matrix for the two voters who can either (V)ote or (A)bstain. This is depicted in Fig. 26 (p. 25).

|             |   | B Supporter    |            |
|-------------|---|----------------|------------|
|             |   | V              | A          |
| A Supporter | V | $1 - c, 1 - c$ | $2 - c, 0$ |
|             | A | $0, 2 - c$     | $1, 1$     |

Figure 26: The Election Game with Two Voters.

Since  $0 < c < 1$ , this game is exactly like the Prisoners' Dilemma: both citizens vote and the candidates tie.

- Here, we need to consider several cases. (Keep in mind that each candidate has an equal number of supporters.) Let  $n_A \leq k$  denote the number of citizens who vote for A and

let  $n_B \leq m$  denote the number of citizens who vote for B. We restrict our attention to the case where  $n_A \geq n_B$  (the other case is symmetric, so there is no need to analyze it separately). We now have to consider several different outcomes with corresponding classes of strategy profiles: (1) the candidates tie with either (a) all  $k$  citizens voting for A or (b) some of them abstaining; (2) some candidate wins either (a) by one vote or (b) by two or more votes. Thus, we have four cases to consider:

- (a)  $n_A = n_B = k$ : Any voting supporter who deviates by abstaining causes his candidate to lose the election and receives a payoff of  $0 < 1 - c$ . Thus, no voting supporter wants to deviate. This profile is a Nash equilibrium.
- (b)  $n_A = n_B < k$ : Any abstaining supporter who deviates by voting causes his candidate to win the election and receives a payoff of  $2 - c > 1$ . Thus, an abstaining supporter wants to deviate. This profile is not Nash equilibrium.
- (c)  $n_A = n_B + 1$  or  $n_B = n_A + 1$ : Any abstaining supporter of the losing candidate who deviates by voting causes his candidate to tie and increases his payoff from 0 to  $1 - c$ . These profiles are not Nash equilibria.
- (d)  $n_A \geq n_B + 2$  or  $n_B \geq n_A + 2$ : Any supporter of the winning candidate who switches from voting to abstaining can increase his payoff from  $2 - c$  to 2. Thus, these profiles cannot be Nash equilibria.

Therefore, this game has a unique Nash equilibrium, in which everybody votes and the candidates tie.

(c) Let's apply very similar logic to this part as well:

- (a)  $n_A = n_B \leq k$ : Any supporter of B who switches from abstaining to voting causes B to win and improves his payoff from 1 to  $2 - c$ . Such a profile cannot be a Nash equilibrium.
- (b)  $n_A = n_B + 1$  and  $n_B = n_A + 1$ , with  $n_A < k$ : Any supporter of the losing candidate can switch from abstaining to voting and cause his candidate to tie, increasing his payoff from 0 to  $1 - c$ . Such a profile cannot be a Nash equilibrium.
- (c)  $n_A = k$  and  $n_B = k + 1$ : Any supporter of A can switch from voting to abstaining and save the cost of voting for a losing candidate, improving his payoff from  $-c$  to 0. Such a profile cannot be a Nash equilibrium.
- (d)  $n_A \geq n_B + 2$  or  $n_B \geq n_A + 2$ : Any supporter of the winning candidate can switch from voting to abstaining and improve his payoff from  $2 - c$  to 2. Such a profile cannot be a Nash equilibrium.

Thus, when  $k < m$ , the game has no Nash equilibrium (in pure strategies).

### 3 Symmetric Games

A useful class of normal form games can be applied in the study of interactions which involve anonymous players. Since the analyst cannot distinguish between the player, it follows that they have the same strategy sets (otherwise the analyst could tell them apart from the different strategies they have available). These games are most often used for 2 player interactions.

DEFINITION 10. A two-player normal form game is **symmetric** if the players' sets of strategies are the same and their payoff functions are such that

$$u_1(s_1, s_2) = u_2(s_2, s_1) \text{ for every } (s_1, s_2) \in S.$$

That is, player 1's payoff from a profile in which he chooses strategy  $s_1$  and his opponent chooses  $s_2$  is the same as player 2's payoff from a profile, in which he chooses  $s_1$  and player 1 chooses  $s_2$ . Note that these do not really have to be equal, it just has to be the case that the outcomes are ordered the same way for each player. (Thus, we're not doing interpersonal comparisons.) Once we have the same ordinal ranking, we can always rescale the appropriate utility function to give the same numbers as the other. Therefore, we continue using the equality but while keeping in mind that it does not have to hold. A generic example, as in Fig. 27 (p. 27) might help. You can probably already see that Prisoners' Dilemma and Stag

|   | A      | B      |
|---|--------|--------|
| A | $w, w$ | $x, y$ |
| B | $y, x$ | $z, z$ |

Figure 27: The Symmetric Game.

Hunt are symmetric while BoS is not. We now define a special solution concept:

DEFINITION 11. A strategy profile  $(s_1^*, s_2^*)$  is a **symmetric Nash equilibrium** if it is a Nash equilibrium and  $s_1^* = s_2^*$ .

Thus, in a symmetric Nash equilibrium, all players choose the same strategy in equilibrium. For example, consider the game in Fig. 28 (p. 27). It has three Nash equilibria in pure strategies:  $(A, A)$ ,  $(C, A)$ , and  $(A, C)$ . Only  $(A, A)$  is symmetric.

|   | A   | B   | C   |
|---|-----|-----|-----|
| A | 1,1 | 2,1 | 4,1 |
| B | 1,2 | 5,5 | 3,6 |
| C | 1,4 | 6,3 | 0,0 |

Figure 28: Another Symmetric Game.

Let's analyze several games where looking for symmetric Nash equilibria make sense.

### 3.1 Heartless New Yorkers

A pedestrian is hit by a taxi (happens quite a bit in NYC). There are  $n$  people in the vicinity of the accident, and each of them has a cell phone. The injured pedestrian is unconscious and requires immediate medical attention, which will be forthcoming if at least one of the  $n$  people calls for help. Simultaneously and independently each of the  $n$  bystanders decides whether to call for help or not. Each bystander obtains  $v$  units of utility if the injured person receives help. Those who call pay a personal cost of  $c < v$ . If no one calls, each bystander receives a utility of 0. Find the symmetric Nash equilibrium of this game. What is the probability no one calls for help in equilibrium?

We begin by noting that there is no symmetric Nash equilibrium in pure strategies: If no bystander calls for help, then one of them can do so and receive a strictly higher payoff of

$v - c > 0$ . If all call for help, then any one can deviate by not calling and receive a strictly higher payoff  $v > v - c$ . (Note that there are  $n$  asymmetric Nash equilibria in pure strategies: the profiles, where exactly one bystander calls for help and none of the others does, are all Nash equilibria. However, the point of the game is that these bystanders are anonymous and do not know each other. Thus, it makes sense to look for a symmetric equilibrium.)

Thus, the symmetric equilibrium, if one exists, should be in mixed strategies. Let  $p$  be the probability that a person does not call for help. Consider bystander  $i$ 's payoff of this mixed strategy profile. If each of the other  $n - 1$  bystanders does not call for help, help will *not* arrive with probability  $p^{n-1}$ , which means that it will be called (by at least one of these bystanders) with probability  $1 - p^{n-1}$ .

What is  $i$  to do? His payoff is  $[p^{n-1}(0) + (1 - p^{n-1})v] = (1 - p^{n-1})v$  if he does not call, and  $v - c$  if he does. From Proposition 1, we must find  $p$  such that the payoffs from his two pure strategies are the same:

$$\begin{aligned}(1 - p^{n-1})v &= v - c \\ p^{n-1} &= c/v \\ p^* &= (c/v)^{\frac{1}{n-1}}\end{aligned}$$

Thus, when all other bystanders play  $p = p^*$ ,  $i$  is indifferent between calling and not calling. This means he can choose any mixture of the two, and in particular, he can choose  $p^*$  as well. Thus, the symmetric mixed strategy Nash equilibrium is the profile where each bystander calls with probability  $1 - p^*$ .

To answer the second question, we compute the probability which equals:

$$p^{*n} = (c/v)^{\frac{n}{n-1}}$$

Since  $n/(n - 1)$  is decreasing in  $n$ , and because  $c/v < 1$ , it follows that the probability that nobody calls is increasing in  $n$ . The unfortunate (but rational) result is that as the number of bystanders goes up, the probability that someone will call for help goes down.<sup>4</sup> Intuitively, the reason for this is that while person  $i$ 's payoff to calling remains the same regardless of the number of bystanders, the payoff to not calling increases as that number goes up, so he becomes less likely to call. This is not surprising. What is surprising, however, is that as the size of the group increases, the probability that at least one person will call for help decreases.

### 3.2 Rock, Paper, Scissors

Two kids play this well-known game. On the count of three, each player simultaneously forms his hand into the shape of either a rock, a piece of paper, or a pair of scissors. If both pick the same shape, the game ends in a tie. Otherwise, one player wins and the other loses according to the following rule: rock beats scissors, scissors beats paper, and paper beats rock. Each obtains a payoff of 1 if he wins,  $-1$  if he loses, and 0 if he ties. Find the Nash equilibria.

We start by the writing down the normal form of this game as shown in Fig. 29 (p. 29).

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<sup>4</sup>There is an infamous case of this dynamic in action: the murder of Kitty Genovese in New York City in 1964, with 38 neighbors looking on without calling the police.

|          |          | Player 2 |          |          |
|----------|----------|----------|----------|----------|
|          |          | <i>R</i> | <i>P</i> | <i>S</i> |
| Player 1 | <i>R</i> | 0, 0     | -1, 1    | 1, -1    |
|          | <i>P</i> | 1, -1    | 0, 0     | -1, 1    |
|          | <i>S</i> | -1, 1    | 1, -1    | 0, 0     |

Figure 29: Rock, Paper, Scissors.

It is immediately obvious that this game has no Nash equilibrium in pure strategies: The player who loses or ties can always switch to another strategy and win. This game is symmetric and we shall look for symmetric mixed strategy equilibria first.

Let  $p$ ,  $q$ , and  $1 - p - q$  be the probability that a player chooses  $R$ ,  $P$ , and  $S$  respectively. We first argue that we must look only at completely mixed strategies (that is, mixed strategies that put positive probability on every available pure strategy). Suppose not, and so  $p_1 = 0$  in some (possibly asymmetric) MSNE. If player 1 never chooses  $R$ , then playing  $P$  is strictly dominated by  $S$  for player 2, and so he will play either  $R$  or  $S$ . However, if player 2 never chooses  $P$ , then  $S$  is strictly dominated by  $R$  for player 1, and so player 1 will choose either  $R$  or  $P$  in equilibrium. However, since player 1 never chooses  $R$ , it follows that he must choose  $P$  with probability 1. But in this case player 2's optimal strategy will be to play  $S$ , to which either  $R$  or  $S$  are better choices than  $P$ . Therefore,  $p_1 = 0$  cannot occur in equilibrium. Similar arguments establish that in any equilibrium, any strategy must be completely mixed.

We now look for a symmetric equilibrium. Player 1's payoff from  $R$  is  $p(0) + q(-1) + (1 - p - q)(1) = 1 - p - 2q$ . His payoff from  $P$  is  $2p + q - 1$ . His payoff from  $S$  is  $q - p$ . In a MSNE, the payoffs from all three pure strategies must be the same, and so

$$1 - p - 2q = 2p + q - 1 = q - p$$

Solving these equalities yields  $p = q = 1/3$ . Thus, whenever player 2 plays the three pure strategies with equal probability, player 1 is indifferent between his pure strategies, and so can play any mixture. In particular, he can play the same mixture as player 2, which would leave player 2 indifferent among his pure strategies. This verifies the first condition in Proposition 1. Because these strategies are completely mixed, we are done. The symmetric Nash equilibrium is  $(1/3, 1/3, 1/3)$ .

Is this the only MSNE? We already know that any mixed strategy profile must consist only of completely mixed strategies in equilibrium. Arguing in a way similar to that for the pure strategies, we can show that there can be no equilibrium in which players put different weights on their pure strategies.

Generally, you should check for MSNE in all combinations. That is, you should check whether there are equilibria, in which one player chooses a pure strategy and the other mixes; equilibria, in which both mix; and equilibria in which neither mixes. Note that the mixtures need not be over the entire strategy spaces, which means you should check every possible subset.

Thus, in a  $2 \times 2$  two-player game, each player has three possible choices: two in pure strategies and one that mixes between them. This yields 9 total combinations to check. Similarly, in a  $3 \times 3$  two-player game, each player has 7 choices: three pure strategies, one completely mixed, and three partially mixed. This means that we must examine 49 combinations! (You can see how this can quickly get out of hand.) Note that in this case, you must check *both* conditions of Proposition 1.

## 4 Strictly Competitive Games

This is a special class of games that is not studied any more as much as it used to be. Nevertheless, it is important to know about them because (a) the results are not difficult, and (b) the results will be useful in later parts of the course.

A strictly competitive game is a two-player game where players have strictly opposed rankings over the outcomes. A good example is MATCHING PENNIES. That is, when comparing various strategy profiles, whenever one player's payoff increases, the other player's payoff decreases. Thus, there is no room for coordination or compromise. More formally,

**DEFINITION 12.** A **two-player strictly competitive game** is a two-player game with the property that, for every two strategy profiles,  $s, s' \in S$ ,

$$u_1(s) > u_1(s') \Leftrightarrow u_2(s) < u_2(s').$$

A special case of strictly competitive games are the zero-sum games where the sum of the two players' payoffs is zero (e.g. MATCHING PENNIES).

|          |          |          |          |
|----------|----------|----------|----------|
|          |          | Player 2 |          |
|          |          | <i>L</i> | <i>R</i> |
| Player 1 | <i>U</i> | 3, 2     | 0, 4     |
|          | <i>D</i> | 6, 1     | 1, 3     |

Figure 30: A Strictly Competitive Game.

Consider the (non-zero-sum) strictly competitive game in Fig. 30 (p. 30). What is the worst-case scenario that player 1 could ever face? This is the case where player 2 chooses *R*, which yields a smaller payoff to player 1 whether he chooses *U* or *D* (he gets  $0 < 3$  if he chooses *U* and  $1 < 6$  if he chooses *D*).

More generally, the *worst* payoff that player  $i$  can get when he plays the (possibly mixed) strategy  $\sigma_i$  is defined by

$$w_i(\sigma_i) \equiv \min_{s_j \in S_j} u_i(s_i, s_j).$$

Which means that we look at all strategies available to player  $j$  to find the one that gives player  $i$  the smallest possible payoff if he plays  $\sigma_i$ . In other words, if player  $i$  chooses  $\sigma_i$ , he is guaranteed to receive a payoff of *at least*  $w_i(\sigma_i)$ . This is the smallest payoff that player 2 can hold player 1 to given player 1's strategy. A security strategy gives player  $i$  the best of the worst. That is, it solves  $\max_{\sigma_i \in \Sigma_i} w_i(\sigma_i)$ :

**DEFINITION 13.** A strategy  $\hat{\sigma}_i \in \Sigma_i$  for player  $i$  is called a **security strategy** if it solves the expression

$$\max_{\sigma_i \in \Sigma_i} \min_{s_j \in S_j} u_i(\sigma_i, s_j),$$

which also represents player  $i$ 's **security payoff**.

Returning to our example in Fig. 30 (p. 30), player 1's security strategy is *D* because given that player 2 is minimizing player 1's payoff by playing *R*, player 1 can maximize it by choosing *D* (because  $1 > 0$ ). Similarly, player 1 can hold player 2 to at most 3 by playing *D*, to which player 2's best response is *R*.

Although there is no general relationship between security strategies and equilibrium strategies, an important exception exists for strictly competitive games.

PROPOSITION 3. *If a strictly competitive game has a Nash equilibrium,  $(\sigma_1^*, \sigma_2^*)$ , then  $\sigma_1^*$  is a security strategy for player 1 and  $\sigma_2^*$  is a security strategy for player 2.*

Note that in our example, the unique Nash equilibrium is  $(D, R)$ , which consists of the exact security strategies we found for both players.

If we look at MATCHING PENNIES, we note that  $w_1(H) = -1$  when  $s_2 = T$ , and  $w_1(T) = -1$  when  $s_2 = H$ . On the other hand  $w_1(1/2) = 0$ , and so player 1's security strategy is to mix between his two actions with equal probability. A symmetric argument establishes the same result for the other player. Again, the unique Nash equilibrium of this game is in security strategies.

## 5 Five Interpretations of Mixed Strategies

See Osborne and Rubinstein's *A Course in Game Theory*, pp. 37-44 for a more detailed treatment of this subject. Here, I only sketch several substantive justifications for mixed strategies.

### 5.1 Deliberate Randomization

The notion of mixed strategy might seem somewhat contrived and counter-intuitive. One (naïve) view is that playing a mixed strategy means that the player deliberately introduces randomness into his behavior. That is, a player who uses a mixed strategy commits to a randomization device which yields the various pure strategies with the probabilities specified by the mixed strategy. After all players have committed in this way, their randomization devices are operated, which produces the strategy profile. Each player then consults his randomization device and implements the pure strategy that it tells him to. This produces the outcome for the game.

This interpretation makes sense for games where players try to outguess each other (e.g. strictly competitive games, poker, and tax audits). However, it has two problems.

First, the notion of mixed strategy equilibrium does not capture the players' motivation to introduce randomness into their behavior. This is usually done in order to influence the behavior of other players. We shall rectify some of this once we start working with extensive form games, in which players move can sequentially.

Second, and perhaps more troubling, in equilibrium a player is indifferent between his mixed strategy and any other mixture of the strategies in the support of his equilibrium mixed strategies. His equilibrium mixed strategy is only one of many strategies that yield the same expected payoff given the other players' equilibrium behavior.

### 5.2 Equilibrium as a Steady State

Osborne (and others) introduce Nash equilibrium as a steady state in an environment in which players act repeatedly and ignore any strategic link that may exist between successive interactions. In this sense, a mixed strategy represents information that players have about past interactions. For example, if 80% of past play by player 1 involved choosing strategy  $A$  and 20% involved choosing strategy  $B$ , then these frequencies form the beliefs each player can form about the future behavior of other players when they are in the role of player 1. Thus, the corresponding belief will be that player 1 plays  $A$  with probability .8 and  $B$

with probability  $\frac{1}{2}$ . In equilibrium, the frequencies will remain constant over time, and each player's strategy is optimal given the steady state beliefs.

### 5.3 Pure Strategies in an Extended Game

Before a player selects an action, he may receive a private signal on which he can base his action. Most importantly, the player may not consciously link the signal with his action (e.g. a player may be in a particular mood which made him choose one strategy over another). This sort of thing will appear random to the other players if they (a) perceive the factors affecting the choice as irrelevant, or (b) find it too difficult or costly to determine the relationship.

The problem with this interpretation is that it is hard to accept the notion that players deliberately make choices depending on factors that do not affect the payoffs. However, since in a mixed strategy equilibrium a player is indifferent among his pure strategies in the support of the mixed strategy, it may make sense to pick one because of mood. (There are other criticisms of this interpretation, see O&R.)

### 5.4 Pure Strategies in a Perturbed Game

Harsanyi introduced another interpretation of mixed strategies, according to which a game is a frequently occurring situation, in which players' preferences are subject to small random perturbations. Like in the previous section, random factors are introduced, but here they affect the payoffs. Each player observes his own preferences but not that of other players. The mixed strategy equilibrium is a summary of the frequencies with which the players choose their actions over time.

Establishing this result requires knowledge of Bayesian Games, which we shall obtain later in the course. Harsanyi's result is so elegant because even if no player makes any effort to use his pure strategies with the required probabilities, the random variations in the payoff functions induce each player to choose the pure strategies with the right frequencies. The equilibrium behavior of other players is such that a player who chooses the uniquely optimal pure strategy for each realization of his payoff function chooses his actions with the frequencies required by his equilibrium mixed strategy.

### 5.5 Beliefs

Other authors prefer to interpret mixed strategies as beliefs. That is, the mixed strategy profile is a profile of beliefs, in which each player's mixed strategy is the common belief of all other players about this player's strategies. Here, each player chooses a single strategy, not a mixed one. An equilibrium is a steady state of beliefs, not actions. This interpretation is the one we used when we defined MSNE in terms of best responses. The problem here is that each player chooses an action that is a best response to equilibrium beliefs. The set of these best responses includes every strategy in the support of the equilibrium mixed strategy (a problem similar to the one in the first interpretation).

## 6 The Fundamental Theorem (Nash, 1950)

Since this theorem is such a central result in game theory, we shall present a somewhat more formal version of it, along with a sketch of the proof. A *finite* game is a game with finite number of players and a finite strategy space. The following theorem due to John Nash

(1950) establishes a very useful result which guarantees that the Nash equilibrium concept provides a solution for every finite game.

**THEOREM 1.** *Every finite game has at least one mixed strategy equilibrium.*

Recall that a pure strategy is a degenerate mixed strategy. This theorem does not assert the existence of an equilibrium with non-degenerate mixing. In other words, every finite game will have at least one equilibrium, in pure or mixed strategies.

The proof requires the idea of best response correspondences we discussed. However, it is moderately technical in the sense that it requires the knowledge of continuity properties of correspondences and some set theory. I will give the outline of the proof here but you should read Gibbons pp. 45-48 for some additional insight.

*Proof.* Recall that player  $i$ 's best response correspondence  $BR_i(\sigma_{-i})$  maps each strategy profile  $\sigma$  to a set of mixed strategies that maximize player  $i$ 's payoff when the other players play  $\sigma_{-i}$ . Let  $r_i = BR_i(\sigma)$  for all  $\sigma \in \Sigma$  denote player  $i$ 's best reaction correspondence. That is, it is the set of best responses for all possible mixed strategy profiles. Define  $r : \Sigma \rightrightarrows \Sigma$  to be the Cartesian product of the  $r_i$ . (That is,  $r$  is the set of all possible combinations of the players best responses.) A *fixed point* of  $r$  is a strategy profile  $\sigma^* \in r(\sigma^*)$  such that, for each player,  $\sigma_i^* \in r_i(\sigma^*)$ . In other words, a fixed point of  $r$  is a Nash equilibrium.

The second step involves showing that  $r$  actually has a fixed point. Kakutani's fixed point theorem establishes four conditions that together are sufficient for  $r$  to have a fixed point:

1.  $\Sigma$  is compact,<sup>5</sup> convex,<sup>6</sup> nonempty subset of a finite-dimensional Euclidean space;<sup>7</sup>
2.  $r(\sigma)$  is nonempty for all  $\sigma$ ;
3.  $r(\sigma)$  is convex for all  $\sigma$ ;
4.  $r$  is upper hemi-continuous.<sup>8</sup>

We must now show that  $\Sigma$  and  $r$  meet the requirements of Kakutani's theorem. Since  $\Sigma_i$  is a simplex of dimension  $\#S_i - 1$  (that is, the number of pure strategies player  $i$  has less 1), it is compact, convex, and nonempty. Since the payoff functions are continuous and defined on compact sets, they attain maxima, which means  $r(\sigma)$  is nonempty for all  $\sigma$ . To see the third case, note that if  $\sigma' \in r(\sigma)$  and  $\sigma'' \in r(\sigma)$  are both best response profiles, then for each player  $i$  and  $\alpha \in (0, 1)$ ,

$$u_i(\alpha\sigma'_i + (1 - \alpha)\sigma''_i, \sigma_{-i}) = \alpha u_i(\sigma'_i, \sigma_{-i}) + (1 - \alpha)u_i(\sigma''_i, \sigma_{-i}),$$

that is, if both  $\sigma'_i$  and  $\sigma''_i$  are best responses for player  $i$  to  $\sigma_{-i}$ , then so is their weighted average. Thus, the third condition is satisfied. The fourth condition requires sequences but the intuition is that if it were violated, then at least one player will have a mixed strategy

<sup>5</sup>Any sequence in  $\Sigma$  has a subsequence that converges to a point in  $\Sigma$ . Alternatively, a compact set is closed and bounded.

<sup>6</sup> $\Sigma$  is convex if every convex combination of any two points in the set is also in the set.

<sup>7</sup>For our purposes, the Euclidean space is the same as  $\mathbb{R}^n$ , i.e. the set of  $n$ -tuples of real numbers.

<sup>8</sup>A correspondence is upper-hemicontinuous at  $x_0$  if every sequence in which  $r(x) \rightarrow x_0$  has a limit which lies in the image set of  $x_0$ . That is, if  $(\sigma^n, \hat{\sigma}^n) \rightarrow (\sigma, \hat{\sigma})$  with  $\hat{\sigma}^n \in r(\sigma^n)$ , then  $\hat{\sigma} \in r(\sigma)$ . This condition is also sometimes referred to as  $r(\cdot)$  having a *closed graph*.

that yields a payoff that is strictly better than the one in the best response correspondence, a contradiction.

Thus, all conditions of Kakutani's fixed point theorem are satisfied, and the best reaction correspondence has a fixed point. Hence, every finite game has at least one Nash equilibrium.  $\square$

Somewhat stronger results have been obtained for other types of games (e.g. games with uncountable number of actions). Generally, if the strategy spaces and payoff functions are well-behaved (that is, strategy sets are nonempty compact subset of a metric space, and payoff functions are continuous), then Nash equilibrium exists. Most often, some games may not have a Nash equilibrium because the payoff functions are discontinuous (and so the best reply correspondences may actually be empty).

Note that some of the games we have analyzed so far do not meet the requirements of the proof (e.g. games with continuous strategy spaces), yet they have Nash equilibria. This means that Nash's Theorem provides sufficient, but not necessary, conditions for the existence of equilibrium. There are many games that do not satisfy the conditions of the Theorem but that have Nash equilibrium solutions.

Now that existence has been established, we want to be able to characterize the equilibrium set. Ideally, we want to have a unique solution, but as we shall see, this is a rare occurrence which happens only under very strong and special conditions. Most games we consider will have more than one equilibrium. In addition, in many games the set of equilibria itself is hard to characterize.